THE HIGH SCHOOL ARITHMETIC,

CONTAINING

ALL THE MATTER USUALLY PRESENTED IN A HIGHER ARITHMETIC, ANALYZED AND PRACTICALLY APPLIED; WITH THE MOST APPROVED MODELS AND ANALYSES.

FOR

Public Schools and Academies.

BY

PHILOTUS DEAN, A. M.

LATE PRINCIPAL AND PROFESSOR OF NATURAL SCIENCES IN THE PITTSBURGH CENTRAL HIGH SCHOOL, AND DIRECTOR OF THE ALLEGHENY OBSERVATORY.

EDITED BY J. P. CAMERON.

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PREFACE.

My name appearing in connection with that of the late Prof. Philotus Dean on the title-page of this work, it is proper for me to account for the same by stating the circumstances connected therewith.

After the death of Prof. Dean, his publishers, together with some of his personal friends, requested that I should prepare for publication his High School Arithmetic, which was then unfinished, and his manuscripts and materials were handed to me for that purpose.

In the performance of this work, it has been my aim to pursue, as nearly as possible, the course indicated in the other Arithmetics of this series. The language of the author, therefore, has been retained, with few changes, as far as the manuscript was complete and ready for publication. Some portions, however, that appeared to be unnecessary in a work of this character, have been omitted, and other changes that seemed essential have been made with due caution, and with a desire to preserve the general plan of the work.

This book has been prepared to complete a thoroughly progressive series of Arithmetics, and is intended to furnish to advanced pupils a very full and comprehensive text-book on the Science of Numbers, embracing all the subjects necessary to a complete and practical arithmetical education.

The exercises presented are numerous and interesting, and have been prepared with special reference to practical utility.

As a well-studied formula of reasoning and recitation is the best means of giving both to thought and its expression the requisite degree of brevity, clearness, and force, the teacher should require from a class a full explanation of every written process, either by following the methods presented in the book, or by using such as he may furnish for his own use.

J. P. CAMERON.

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CHAPTER I.

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DEFINITIONS.

Article 1. **Mathematics** is the science of quantity.

Art. 2. The fundamental branches of mathematics are Arithmetic, Algebra, and Geometry.

Art. 3. **Arithmetic** is the science of numbers and the art of computation.

A **number** is a definite expression of quantity.

Quantity is that which is capable of measurement.

A **unit** is one. The **unit** of a **number** is one of that number. Thus, the unit of 5 is 1; the unit of 5 feet is 1 foot. It is sometimes called *unity*.

Art. 4. In reference to its ultimate nature, quantity may be *magnitude* or *multitude*.

Magnitude, in its primary signification, is extent in space. In this sense it is of three kinds, viz.: *length*, *surface*, and *solidity*.

Magnitude in its secondary signification, is quantity of anything concerning which the question "How much?" can be properly asked, such as *space*, *duration*, *force*, *value*, &c.

Multitude is quantity of separate things. It is that concerning which the question "How many?" can be properly asked.

Art. 5. **Measuring** a quantity is applying to it a unit of the same kind, to find how many times the unit is contained in that quantity. A **measure** is a unit used in measuring.

In measuring magnitudes a certain quantity is generally assumed as a unit, to measure other quantities of the same kind; as a pound, for weight; an hour, for time; a yard, for length; a gallon, for capacity, &c. Such units may be called *artificial units.*

In measuring multitude of any kind of thing, the units are the individuals of that kind, without reference to their being equal or unequal; as a tree, a cloud, a bird, a hill, a man, &c. Such units may be called *natural units.*

Art. 6. In reference to the wholeness of their unit, numbers are of three classes, viz.: *integral, fractional,* and *mixed.*

An **integer** is any whole number; as, 1, 5, 7, 9, &c.

A **fraction** is a number expressing one or more of the equal parts of a unit, or of a quantity; as $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{8}$, &c.

A **mixed** number is a number composed of an integer and a fraction; as, $5\frac{1}{3}$, $10\frac{3}{8}$, 7.5, &c.

Art. 7. In reference to their application, numbers are of two classes, viz.: *concrete,* and *abstract.*

A **concrete number** is an expression of a particular kind of quantity; as, 5 men, 8 dollars, 20 houses, &c.

An **abstract number** is an expression of quantity without reference to its kind; as, 5, 8, 20, &c.

Art. 8. In reference to the measuring quality of their units, concrete numbers may be *denominate,* or not.

A concrete number is **denominate** when its units are measures of the magnitude of the thing mentioned. Thus, *five feet* is a denominate, because *a foot* is a unit which measures quantity of length. For similar reasons, *seven pounds* is a denominate number in weight, or force; *nine*

hours, in time; and *ten dollars*, in money, or value. Each kind of such units is called a **denomination.**

A concrete number is *not denominate* when its units are not measures of the magnitude of the thing mentioned, but merely of its multitude. Thus, *two birds, seven hills, ten houses, three thoughts*, &c.

Art. 9. In reference to the number of denominations in them, denominate numbers are *simple* or *compound.*

A **simple** denominate number is expressed in one denomination; as, 5 *feet.*

A **compound** denominate number is expressed in more than one denomination; as, 5 feet 7 inches.

Art. 10. Numbers, compared in reference to the nature of their quantities, are either *like* or *unlike.*

Like numbers are those which express the same kind of quantity. Thus, 5 and 7 are like numbers, because both have abstract units; 5 *dollars* and 7 *dollars* are like numbers, because both have concrete units of the same nature.

Unlike numbers are those which express different kinds of quantity. Thus, 5 *dollars* and 7 *miles* are unlike numbers.

Art. 11. In reference to the method of computing, Arithmetic may be *Mental*, or *Written.*

In **Mental Arithmetic** computations are performed without the aid of written characters. It is also called *Intellectual Arithmetic.*

In **Written Arithmetic** computations are performed with the aid of written characters.

Art. 12. In reference to its subjects, Arithmetic is either *pure* and *theoretical*, or *applied* and *practical.*

Pure, or **theoretical Arithmetic** treats of the nature and principles of numbers, and of operations with them, without regard to their application.

Applied, or **practical Arithmetic** treats of computations in science, arts, and business.

Art. 13. A **rule,** in Arithmetic, is a statement of the mode of computing.

An **example** is a statement of quantities to be used in computing according to a certain rule. It is óften called a *question.*

A **solution** is the act of computing the unknown or required quantities in a question, from those which are known.

An **operation** is that which is done to accomplish a solution.

A **sign** is a character which indicates either a relation between numbers, or an operation to be performed with them.

A **problem** is a question proposed for solution. *To solve* a problem is to compute the required quantity.

A **demonstration** is a process of reasoning which proves a truth.

A **proposition** is a statement of a truth to be demonstrated. It is also called a *theorem.*

A **corollary** is an inference from a proposition.

An **analysis** is a solution whose steps are accompanied with a statement of the reasons for them.

An **axiom** is a self-evident truth. All the processes of mathematics are founded upon axioms. The following are some of them.

Axioms.

1. The whole of a quantity is greater than a part of it.
2. The whole of a quantity is equal to the sum of all its parts.
3. If equal quantities are equally increased, the sums are equal.
4. If equal quantities are equally decreased, the remainders are equal.
5. If equal quantities are unequally increased, the sums are unequal.
6. If equal quantities are unequally decreased, the remainders are unequal.

7. Quantities which are equal to the same quantity, are equal to each other.
8. Like parts of equal quantities are equal.
9. Like powers and roots of equal quantities are equal.

Art. 14. In Written Arithmetic there are six operations on which all others depend, viz.: *Notation, Numeration, Addition, Subtraction, Multiplication,* and *Division.* The last four are frequently called *the four fundamental rules of Arithmetic,* because all computations must use one or more of them.

Synopses of Mathematics and Number.

Mathematics.

BRANCHES.
- **Arithmetic.**
  - With respect to manner of operating
    - Mental.
    - Written
  - With respect to application.
    - Pure.
    - Applied.
- **Algebra.**
- **Geometry.**

Numbers.
- With respect to wholeness of unit.
  - Integral.
  - Fractional.
  - Mixed.
- With respect to application
  - Concrete.
  - Abstract.
    - With respect to quality of unit.
      - Denominate.
        - With respect to number of denominations.
          - Simple.
          - Compound.
      - Not Denominate.
- With respect to kind of quantity.
  - Like.
  - Unlike.

CHAPTER II.

NOTATION.

Art. 15. Notation is the art of expressing numbers and their relations by symbols.

The notation of numbers employs *letters* and *figures.*

The notation of the relation of numbers, employs signs.

Art. 16. There are two systems of notation in common use—the Roman and the Arabic.

NOTE.—The ancient Greeks and Romans used the letters of their alphabet for symbols of number. Most modern nations still use the Roman symbols for some of the purposes of numbering; but for computations they use a set of symbols introduced into Europe by the Arabians. The Arabic Notation is supposed to have been introduced into Europe at the time of the Saracenic invasion, between the seventh and eleventh centuries of the Christian era. It is probable that the system originated in Hindoostan, where it appears to have been in use more than 2000 years. Hence it is sometimes called the Indian Notation. It is, also, sometimes called the Decimal Notation, (from the Latin *decem, ten*) because it uses ten symbols, and each order of value is ten times the next lower order.

ROMAN NOTATION.

Art. 17. The symbols of the Roman Notation are seven capital letters.

	I	V	X	L	C	D	M
Values,	one,	five,	ten,	fifty,	one hundred,	five hundred,	one thousand.

The Roman Notation is founded upon the following principles:—

1ST. Repeating a letter repeats its value. V, L, and D are not repeated.

2D. If a letter be placed at the left of one of greater value, its value is to be taken from that of the greater. Thus, IV denotes four, IX nine, XL forty.

3D. If a letter be placed at the right of one of greater value, its value is to be united to that of the greater. Thus, VI denotes six, XI eleven, LX sixty.

4TH. If a letter be placed between two denoting greater values, its value is to be taken from the united value of the other two. Thus, XIV denotes fourteen, XIX nineteen.

5TH. A line over a letter increases its value a thousandfold. Thus, $\overline{\text{V}}$ denotes five thousand, $\overline{\text{X}}$ ten thousand.

TABLE OF ROMAN NOTATION.

Symbol.	Signification.	Symbol.	Signification.	Symbol.	Signification.
I.	One.	XXI.	Twenty-one.	CC.	Two hundred.
II.	Two.	XXII.	Twenty-two.	CCC.	Three hundred.
III.	Three.	XXIII.	Twenty-three.	CCCC.	Four hundred.
IV.	Four.	XXIV.	Twenty-four.	D, or IↃ.	Five hundred.
V.	Five.	XXV.	Twenty-five.	DC.	Six hundred.
VI.	Six.	XXVI.	Twenty-six.	DCC.	Seven hundred.
VII.	Seven.	XXVII.	Twenty-seven.	DCCC.	Eight hundred.
VIII.	Eight.	XXVIII.	Twenty-eight.	DCCCC.	Nine hundred.
IX.	Nine.	XXIX.	Twenty-nine.	M, or CIↃ.	One thousand.
X.	Ten.	XXX.	Thirty.	MM.	Two thousand.
XI.	Eleven.	XXXI.	Thirty-one.	MMM.	Three thousand.
XII.	Twelve.	XL.	Forty.	MMMM.	Four thousand.
XIII.	Thirteen.	L.	Fifty.	$\overline{\text{V}}$, or IↃↃ.	Five thousand.
XIV.	Fourteen.	LX.	Sixty.	$\overline{\text{L}}$, or IↃↃↃ.	Fifty thous'd.
XV.	Fifteen.	LXX.	Seventy.	$\overline{\text{X}}$, or CCIↃↃ.	Ten thous'd.
XVI.	Sixteen.	LXXX.	Eighty.	$\overline{\text{C}}$, or CCCIↃↃↃ.	A hundr'd thousand.
XVII.	Seventeen	XC.	Ninety.		
XVIII.	Eighteen.	C.	One hundred.	MDCCCLXVIII.	One thousand, eight hundred and sixty-eight.
XIX.	Nineteen.	CI.	One h'd one.		
XX.	Twenty.	CX.	One h'd ten.		

NOTE.—The Roman Notation is commonly used in numbering chapters, sections, books, and public documents.

EXAMPLES.

Express the following numbers by the Roman Notation.

1. Thirty-nine.
2. Fifty-seven.
3. Seventy-eight.
4. Ninety-three.
5. Eighty-six.
6. One hundred twenty.
7. Three hundred eleven.
8. Five hundred thirteen.
9. Seven hundred thirteen.
10. Nine hundred nineteen.
11. One thousand six hundred sixty-one.
12. Four thousand eight hundred eighty-eight.
13. Two hundred thousand forty-four.
14. The present year of the Christian era.

ARABIC NOTATION.

Art. 18. The symbols of the Arabic Notation are ten figures, viz.:—

1 2 3 4 5 6 7 8 9 0
one, two, three, four, five, six, seven, eight, nine, naught.

The first nine of these figures are called *significant figures*, because they signify some number. They are also called *digits*. The figure, 0, *naught*, is also called *cipher* and *zero*, because when alone it expresses no number.

Art. 19. To express more than nine units, two or more figures must be combined, this is done on the following

Principle. *A figure at the left of units expresses* tens; *at the left of tens*, hundreds; *at the left of hundreds*, thousands; *at the left of thousands*, ten thousands; &c. *In general, a figure expresses a ten times greater quantity every time it is placed at the left of one more figure.*

Thus, *ten* is *one ten no units*, and is written with a 1 at the left of a 0; *twenty* is *two tens no units*, and is written with a 2 at the left of a 0, &c. Therefore 10 is *ten;* 20, *twenty;* 30, *thirty;* 40, *forty;* 50, *fifty;* 60, *sixty;* 70, *seventy;* 80, *eighty;* 90, *ninety*.

All other numbers less than a hundred are expressed by placing a significant figure in the units' place. Thus, 11 is

one ten one unit, or *eleven;* 24 is *two tens four units*, or *twenty-four;* &c.

Hundreds are expressed by figures in the third place toward the left. Hence 1 at the left of two 0's signifies *one hundred no tens no units*, or, simply, *one hundred.* Therefore, 100 is *one hundred;* 200, *two hundred;* 300, *three hundred;* 400, *four hundred;* 500, *five hundred;* &c.

Thousands are expressed by figures in the fourth place toward the left. Thus, 1000 is *one thousand;* 2000, *two thousand;* &c.

Tens of thousands are expressed by figures in the fifth place toward the left. Thus, 10000 is *ten thousand;* 20000, *twenty thousand;* &c.

Hundreds of thousands are expressed by figures in the sixth place toward the left. Thus, 100000 is *one hundred thousand;* 200000, *two hundred thousand;* &c.

Millions are expressed by figures in the seventh place toward the left. Thus, 1000000 is *one million;* 2000000, *two million;* &c.

Hence, if we consider *a ten, a hundred, a thousand*, &c., as units of different values, it is plain that ten units of any order of value make one of the next higher order; that is, make *a unit of ten times a greater value.* Thus :—

Ten *units*	make one *ten.*
Ten *tens*	make one *hundred.*
Ten *hundreds*	make one *thousand.*
Ten *thousands*	make one *ten-thousand.*
&c.	&c.

Art. 20. The place of a figure is its position in a number, as determined by reckoning the number of figures it is removed from the units of that number.

The value of a figure is its power to express quantity.

Figures have two kinds of value, viz.: *simple* and *local.*

The simple value of a figure is the quantity which it expresses when it is alone.

The local value of a figure is the quantity which it expresses by occupying a place in a number.

In the units' place of a number the simple and local value of a figure are the same.

NUMERATION.

Art. 21. Numeration is the art of reading numbers expressed by figures.

To assist in reading and writing numbers in the Arabic Notation, the figures are arranged in groups, called periods, beginning at the place of units. There are two methods of grouping in use, viz.: the *French* and the *English.*

French Method of Numeration.

Art. 22. In the French method of numeration three figures form a period. The right-hand figure of every period is the *units* of that period, the middle figure is its *tens,* and the left-hand figure is its *hundreds.*

The names of the periods from right to left, are:—

1st.	Units.	14th.	Duodecillions.
2d.	Thousands.	15th.	Tertio-decillions.
3d.	Millions.	16th.	Quarto-decillions.
4th.	Billions.	17th.	Quinto-decillions.
5th.	Trillions.	18th.	Sexto-decillions.
6th.	Quadrillions.	19th.	Septo-decillions.
7th.	Quintillions.	20th.	Octo-decillions.
8th.	Sextillions.	21st.	Nono-decillions.
9th.	Septillions.	22d.	Vigillions.
10th.	Octillions.	23d.	Primo-vigillions.
11th.	Nonillions.	24th.	Secundo-vigillions.
12th.	Decillions.	32d.	Trigillions.
13th.	Undecillions.	42d.	Quadragillions.

&c.

Illustration of French Numeration.

Hundreds of Quintillions.	Tens of Quintillions.	Quintillions.	Hundreds of Quadrillions.	Tens of Quadrillions.	Quadrillions.	Hundreds of Trillions.	Tens of Trillions.	Trillions.	Hundreds of Billions.	Tens of Billions.	Billions.	Hundreds of Millions.	Tens of Millions.	Millions.	Hundreds of Thousands.	Tens of Thousands.	Thousands.	Hundreds.	Tens.	Units.
7	8	4,	2	9	6,	5	1	3,	8	5	0,	3	7	5,	4	0	0,	6	9	8.
7th period. Quintillions.			6th period. Quadrillions.			5th period. Trillions.			4th period. Billions.			3d period. Millions.			2d period. Thousands.			1st period. Units.		

This number is read thus:—*Seven hundred eighty-four quintillions, two hundred ninety-six quadrillions, five hundred thirteen trillions, eight hundred fifty billions, three hundred seventy-five millions, four hundred thousand, six hundred ninety-eight.*

Art. 23. To read a simple whole number.

Rule. *Beginning at units, mark off the figures into periods of three, toward the left, naming the periods as you proceed. Then, beginning at the left, read and name each period in order, omitting the names of those filled with ciphers, and also the name of the units' period.*

Note.—It is best to omit the word *and* in reading the figures.

Examples in Numeration.

5	13625	2751428	40230041
95	24704	5017043	36004263
905	80530	8270416	78400030
950	731026	9871605	823140100
1234	117600	21716200	703023461
5208	200431	83000513	8170062004
7462	500505	68213000	82403100627

72104536821468465002406

Art. 24. To write a simple whole number.

Rule. *First write the number of the highest period, then that of each period in order toward the right, filling with ciphers the places and periods not expressed.*

Examples in Notation.

1. Three thousand two hundred fifty-seven.
2. Thirty thousand eighty.
3. Ninety thousand nine hundred eighty.
4. Fifty-six millions fifty-six thousand fifty-six.
5. One hundred two millions one hundred one.
6. One trillion ten millions one hundred.
7. Six quadrillions sixty-six billions six hundred sixty thousand.
8. Fifteen quintillions eleven billions one hundred.
9. Twenty sextillions two hundred trillions ten.
10. Seventy octillions six hundred quadrillions forty-six.

Note.—No pupil should be permitted to pass this subject till he can write *any number dictated by the teacher or class,* and afterwards point it off and read it.

English Method of Numeration.

Art. 25. In the English method of numeration six figures form a period. The right-hand figure of every period is its *units;* the next left is its *tens;* the next, its *hundreds;* the next, its *thousands;* the next, its *tens of thousands;* the next, its *hundreds of thousands.* Hence this method has no *period of thousands,* but the periods are, from right to left, *units, millions, billions,* &c., as in the French method.

Illustration of English Numeration.

of Trillions						of Billions						of Millions						of Units					
Hundreds of Thousands.	Tens of Thousands.	Thousands.	Hundreds.	Tens.	Units.	Hundreds of Thousands.	Tens of Thousands.	Thousands.	Hundreds.	Tens.	Units.	Hundreds of Thousands.	Tens of Thousands.	Thousands.	Hundreds.	Tens.	Units.	Hundreds of Thousands.	Tens of Thousands.	Thousands.	Hundreds.	Tens.	Units.
3	0	7	4	5	0,	0	2	5	8	9	2,	6	5	4	8	0	7,	2	2	9	3	3	4

This number, thus grouped, is read as follows:—

Three hundred seven thousand four hundred fifty trillions; twenty-five thousand eight hundred ninety-two billions; six hundred fifty-four thousand eight hundred seven millions; two hundred twenty-nine thousand three hundred thirty-four.

Examples for Practice.

Point off, and read in the English method

1. 30206954008.
2. 900024080095.
3. 60119000401001 1.
4. 520000178201010 6.
5. 1901800170001600001504.
6. 988977686657554644353324221300.
7. 12233344445555566666667777777778888.
8. 21031210430610243242132432143 25.
9. 3203123456789107000432102432 4.
10. 678940023450003214325143210 24.
11. 987654321123456789000000 1447316.

SCALES OF NOTATION.*

Art. 26. A notation which has a distinct figure for every possible number would be too burdensome for memory or use.

A notation whose figures have local values, that is, express different quantities according to their place in a number, is said to have *a scale.*

The **scale** of a notation is its law of local value. Thus, the Arabic notation has a decimal scale, because the local value of a figure is tenfold greater than in the next lower place. The decimal scale may be expressed thus:—

10 units make 1 ten.
10 tens make 1 hundred.
10 hundreds make 1 thousand.
10 thousands make 1 ten-thousand.
&c.

In explaining scales of notation, local values are sometimes called *orders of units:* that is, the lowest local value is called the *first order of units;* the next higher, *the second order of units,* &c.

Art. 27. The **radix,** or **ratio,** of a scale of notation is the number of units of one order which make a unit of the next higher.

PRINCIPLE. *Any system of notation which has a scale must have as many symbols as there are units in its radix. One of these symbols must signify* naught, *and another,* one.

A **binary system** may be constructed with the radix *two,* and the symbols 0 and 1. In such a system, 1 in the second

* This subject may be omitted by young and inexperienced classes. It is best studied after some skill and experience have been acquired, and it is, therefore, more appropriately taken on going a second time through the book, though it is legitimately treated of here under Notation.

place, that is, the expression 10 would signify *two*; 11, *three*; 100, *four*; 101, *five*; 110, *six*; 111, *seven*; 1000, *eight*; 1001, *nine*; 1010, *ten*, &c.

The binary scale may be expressed thus:—

2 ones make 1 *two*.
2 twos make 1 *four*.
2 fours make 1 *eight*.
&c.

A **ternary system** may be constructed with the **radix** *three*, and the symbols 0, 1, and 2. In such a system, 1 in the second place, that is, the expression 10 would signify *three*; 11, *four*; 12, *five*; 20, *six*; 21, *seven*; 22, *eight*; 100, *nine*; 101, *ten*; &c.

The ternary scale may be expressed thus:—

3 ones make 1 *three*.
3 threes make 1 *nine*.
3 nines make 1 *twenty-seven*.
&c.

In a **quaternary** system, with radix *four*, and symbols 0, 1, 2, 3, the expression 10 is *four*; 11, *five*; 12, *six*; 13, *seven*; 20, *eight*; &c.

In a **quinary** system, with radix *five*, and symbols 0, 1, 2, 3, 4, the expression 10 is *five*; 11, *six*; 12, *seven*; 13, *eight*; 14, *nine*; 20, *ten*; 21, *eleven*; 22, *twelve*, &c.

In a **senary** system, with radix *six*, and symbols 0, 1, 2, 3, 4, 5, the expression 10 is *six*; 11, *seven*; 12, *eight*; 20, *twelve*, &c.

In a **septenary** system, with radix *seven*, and symbols 0, 1, 2, 3, 4, 5, 6, the expression 10 is *seven*; 20, *fourteen*; 30, *twenty-one*; &c.

In an **octary** system, with radix *eight*, and symbols 0, 1, 2, 3, 4, 5, 6, 7, the expression 10 is *eight*; 20, *sixteen*; 30, *twenty-four*; &c.

In a **nonary** system, with radix *nine*, and symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, the expression 10 is *nine*; 20 *eighteen*; &c.

The **denary**, or *decimal* system, with radix *ten*, is in common use, and has already been explained.

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In an **undenary,** or *undecimal* system, with **radix** *eleven,* and symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, *t,* the expression 10 is *eleven;* 11, *twelve;* 20, *twenty-two;* 30, *thirty-three; t* 0, *one hundred and ten,* being eleven times ten; &c.

In a **duodenary,** or *duodecimal* system, with **radix** *twelve,* and symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, *t, e,* the expression 10 is *twelve; t* 0 *one hundred and twenty; e* 0, *one hundred and thirty-two;* &c.

It is, therefore, possible to construct a system of notation with any scale whatever, provided that enough symbols are invented to express numbers less than the radix. A *sexagesimal* system, with radix *sixty,* was invented about the second century of the Christian Era. The principal objection to its use was that it required sixty symbols, too great a number for convenience. The decimal system, now generally used, has the advantages of fewness and simplicity of symbols, readiness of computation with *tens,* perfect precision, and great scope of expression.

Art. 28. The reading of numbers, expressed in notations having other scales than the decimal, would require the invention of new names and new styles of grouping, if we should follow the analogy of the decimal numeration. Otherwise we should have to read a number by pronouncing successively the figure in each *place,* with its local value.

ILLUSTRATION IN THE BINARY NOTATION.

four thousand ninety-six	two thousand forty-eight	one thousand twenty-four	five hundred twelve	two hundred fifty-six	one hundred twenty-eight	sixty-four	thirty-two	sixteen	eight	four	two	one
1	1	0	1	0	1	0	1	1	0	1	0	1

If we retain the present decimal names of numbers, and attempt to read the foregoing number, it would be pronounced as follows:—

One four-thousand-ninety-six; one two-thousand-forty-eight; no one-thousand-twenty-four; one five-hundred-twelve; no two-

hundred-fifty-six; one one-hundred-twenty-eight; no sixty-four; one thirty-two; one sixteen; no eight; one four; no two; and one.

On making similar experiments with the other notations, we shall perceive the great convenience of the decimal system, and the almost insuperable difficulties which would lie in the way of attempting to substitute any other for it. All civilized nations use it, and it will probably become and continue to be the world's notation to the end of human affairs.

NOTE.—For the methods of converting numbers of one system into their value in another, see Arts. 74 and 75.

NOTATION OF THE RELATIONS OF NUMBERS.

Art. 29. A **formula** is an expression of quantities and their relations in a problem.

An **equation** is an expression of the equality of two quantities.

The **sign of equality** is two equal and parallel lines, placed in the line of writing, thus, $=$. It signifies that the expressions between which it is placed are equal. Thus, 10 *units* $=$ 1 *ten* is an equation.

The **first member** of an equation is the expression which precedes the sign of equality.

The **second member** of an equation is the expression which follows the sign of equality.

The **sign of inequality** is made by two short lines that meet so that their opening is toward the greater quantity. Thus, $8 > 5$ signifies that 8 is a greater quantity than 5. Again, $5 < 8$ signifies that 5 is a less quantity than 8. The sign $>$ is called the *sign of greater inequality:* $<$ is called the *sign of less inequality.*

NOTE.—The symbols of the notations of the other relations of numbers, and of operations with them, such as *variation, addition, subtraction, multiplication, division,* and of the application of these operations to *involution, evolution, ratio, proportion, percentage,* and the symbols of denominate units will be given in connection with their appropriate topics.

1. Synopsis of Notation.

- **Notation**
  - Of numbers.
    - As to kinds used.
      - Roman.
      - Arabic.
      - Ancient Greek.
      - &c.
    - As to scales.
      - Binary.
      - Ternary.
      - Quaternary.
      - Quinary.
      - Senary.
      - Septenary.
      - Octary.
      - Nonary.
      - **Denary,** or **Decimal.**
      - Undenary.
      - Duodenary.
      - &c.
  - Of the relations of Numbers.
    - As to value.
      - Equality.
      - Inequality.
      - Variability.
    - As to operation.
      - Addition.
      - Subtraction.
      - Multiplication.
      - Division.
  - Of denominate units.

2. Synopsis of Numeration.

- **Numeration.**
  - French.
  - English.

CHAPTER III.

ADDITION OF SIMPLE WHOLE NUMBERS.

Art. 30. Addition is the process of uniting two or more numbers, so as to form one number, which is called their *Sum* or *Amount.*

Art. 31. The **Sign of Addition** is $+$, and is called *plus.* When placed between two numbers, it denotes they are to be added together. Thus, $8 + 5 = 13$, is read *eight plus five equals thirteen.*

Art. 32. Only *like numbers* can be added.

Two or more kinds of quantity cannot make a single number expressing only one kind. Thus, 4 units and 5 tens do not amount either to 9 units or 9 tens. But, if we consider the 5 tens as 50 units, then 4 units and 50 units can be added, making 54 units. So, also, 4 *feet* and 5 *inches* do not amount either to 9 feet or 9 inches. But, if we express 4 feet as 48 inches, the numbers 48 inches and 5 inches can be added, making 53 inches.

Art. 33. The addition of numbers consisting of several orders of units is based upon the following principles:—

I. *The sum of numbers is equal to the sum of their corresponding parts.*

II. *Only units of the same order can be added.*

III. *When the number of units of any order requires but one figure to express it, that figure belongs to that order.*

IV. *When the number of units of any order requires two or more figures to express it, the right-hand figure belongs to that order, the next left-hand figure belongs to the next higher order, and so on.*

Ex. 1. Find the sum of 576, 483, and 697.

WRITTEN PROCESS.
5 7 6
4 8 3
6 9 7
1 7 5 6

EXPLANATION.

Because units of the same kind only can be added, we write the numbers so that units of the same kind shall stand in the same column. The sum of the units' column is 16 *units*, equal to 1 *ten* and 6 *units*. The 6 units are written under the units' column, and the 1 ten is added to the tens' column, making its sum 25 *tens*, equal to 2 *hundreds* and 5 *tens*. The 5 tens are written under the tens column, and the 2 hundreds are added to the hundreds' column, making 17 *hundreds*, equal to 1 *thousand* and 7 *hundreds*. The 7 hundreds are written under the hundreds' column, and the 1 thousand, in thousands' place, making the entire sum 1756. The process of adding the left-hand figure of any sum to the next column is called *carrying*.

In adding it is best not to name each figure, but results only, according to the following

MENTAL AND RECITATION MODEL.

7, 10, 16. Write the 6, and carry the 1:—10, 18, 25. Write the 5, and carry the 2:—8, 12, 17. Write 17.

Rule. *Write the numbers so that their units may form one column, their tens another, &c., and draw a line under them.*

First find the sum of the units. If it is less than ten, write it under the column. If it is ten, or more, write its right-hand figure under the column, and add the other figure or figures to the next column.

Do this with each column successively to the last, of which write the whole sum.

METHODS OF PROOF.

I. Begin with the units, and add each column successively in the direction opposite to that in which it was first added. If the work is correctly done in both cases, the two results are equal.

II. Add together the figures of the numbers added, dropping from the reckoning every *nine* as fast as formed. Do the same with the amount. The excess over nine in the *amount* should equal that in the numbers added.

Ex. 2. Find the sum of 2537, 3845, 2146, and 3720.

WRITTEN PROCESS.		
2 5 3 7	Excess	8
3 8 4 5	"	2
2 1 4 6	"	4
3 7 2 0	"	3
1 2 2 4 8		8

EXPLANATION.

Adding the figures of 2537, 2+5+3=10; drop 9, leaving 1; 1+7=8, excess over 9. In 3845, 3+8=11; drop 9, leaving 2; 2+4+5=11; drop 9, leaving 2, excess over 9. In the same way 4 is the excess in 2146, and 3 is the excess in 3720. The sum of these excesses is 17; drop 9, leaving 8, excess over 9 in the numbers added. In the amount 1+2+2+4=9; drop it, and 8 remains as excess over 9. This is the same excess as in the numbers added. *We need not notice the excess in each number, but merely add all its figures in any order, dropping nine as fast as formed.*

III. If the operator is acquainted with subtraction, he can prove the work by *casting out* elevens, thus:—Beginning at the right of each number added, add the *alternate* figures, dropping *eleven* as fast as formed. Do the same with the other figures, and subtract the last result from the former, increased by 11 if necessary. Then cast out the elevens from these excesses, and from the amount. The excess in the amount should equal that in the numbers added.

ILLUSTRATION.

In Ex. 2, 7+5=12; drop 11, leaving 1; then 3+2=5; excess 5; then 5 from 1 (+11=12) = 7 excess over 11 in 2537. In the same way we find 6 excess in 3845, 1 excess in 2146, and 2 excess in 3720, making 7+6+1+2=16 excess; drop 11, leaving 5 excess in the numbers added. In the amount, 8+2+1=11; drop it; then 4+2 =6; 6 from 0(+11)=5, the same excess as in the numbers added.

NOTE.—No methods of proof give absolute certainty of correct results.

EXAMPLES FOR PRACTICE.

(3)	(4)	(5)	(6)
6385	7456983	76	60070
4479	123456	235	8009
8002	78925	4682	100
5634	4807	72847	20030
9753	684	496004	4005
34253			

7. Find the sum of the numbers from 210 to 225 inclusive.

8. From 4541 to 4556 inclusive.

9. From 6773 to 6788 inclusive.

10. From 8805 to 8820 inclusive.

11. $1006 + 85 + 273 + 824 + 9 + 25 =$ how many?

12. $73 + 1427 + 876 + 493 + 62 + 6 =$ how many?

13. Add *fifty-seven billions two hundred nine thousand and twenty; nine hundred five millions, nine hundred ninety thousand, eight hundred eighty-seven, and nine hundred forty-two billions ninety-three millions eight hundred thousand and ninety-three.*

14. Find the sum of *five hundred thirty-seven, twenty-six thousand four hundred eighty-five, seven million five hundred ninety-seven thousand three hundred eighteen, eight hundred sixty millions four hundred thousand two hundred; ninety billions eight millions six hundred thousand,* and *seven trillions nine hundred ninety-eight billions eight hundred seventy-seven millions six hundred sixty-five thousand five hundred forty-four.*

15. If in 1850, Pennsylvania raised 15367691 bushels of wheat, Ohio 14487351 bushels, New York 13121498 bushels, Illinois 9414575 bushels, Indiana 6214458 bushels, Virginia 11212616 bushels, Michigan 4925889 bushels, and Maryland 4494680 bushels, how many bushels did all these States raise that year?

16. If the Caspian Sea contains 145123 square miles, Lake Superior 31534, Michigan 23155, Huron 23123, Erie 7799, Ontario 6901, Winnipeg 6495, and Great Salt Lake 1875, what is the area of all these bodies of water?

17. A began the year 1861 seventy-five thousand nine hundred and eighty-four dollars in debt, and at the beginning of 1865 he was worth, free of all debts, 864575 dollars. How much did he clear in those four years?

18. B began business with 49263 dollars, and ended by being 37877 dollars in debt after giving up all his property. *How much* did he lose?

19. A ship started from a place 2799 miles north of the Equator, and sailed due south to a place 2553 miles south of the Equator. How many miles did she sail?

20. A thermometer showed 37 degrees below zero one winter, and 102 degrees above zero the next summer. How much did it vary?

21. If Cyrus took Babylon 538 years before Christ, how many years ago was that event?

22. Find the sum of two hundred fifty-nine thousand six hundred thirty-eight, seventy-one thousand five, nineteen thousand ninety, six hundred four thousand two, three hundred twelve thousand eight hundred, fifty-seven thousand, twenty-five thousand sixty-six, and nine hundred three.

Ex. 23. Find the sum of 2345, 3761, 4983, and 5736, by adding more than one column at a time.

WRITTEN PROCESS.	EXPLANATION.
2 3 4 5 3 7 6 1 4 9 8 3 5 7 3 6 ――――― 1 6 8 2 5	If we take two columns at a time, we may proceed thus:—36 and 3 are 39, and 80 are 119, and 1 is 120, 60 are 180, and 5 are 185, and 40 are 225. Write the 25 and carry the 2 hundreds to 57 hundred, making 59 hundred; and 40 are 99, and 9 are 108, and 30 are 138, and 7 are 145, and 20 are 165, and 3 are 168 hundreds. Write the whole.

If we wished to add three columns at a time, we would proceed thus:—736 and 3 are 739, and 80 are 819, and 900 are 1719, and 1 is 1720, and 60 are 1780, and 700 are 2480, and 5 are 2485, and 40 are 2525, and 300 are 2825, write the 825, and carry the 2 thousands; &c.

(24)	(25)	(26)	(27)	(28)
6385	7456	3594	4963	804
4479	983	4672	745	9725
8002	2548	7231	5278	576
5634	6007	6327	3826	342
7953	765	5846	7482	498

(29)	(30)	(31)	(32)	(33)
35791	24680	20304	60070	76542
42863	7531	5060	8009	13890
76135	86420	70809	100	98765
93647	5678	1020	20030	43210
64208	63957	30405	4005	12345
12345	9085	4050	320	78654

34. A manufacturer's weekly expenditures for a year were, respectively, 739, 546, 800, 406, 628, 942, 1312, 825, 1536, 597, 1200, 1031, 304, 648, 846, 2037, 1991, 1404, 1655, 2577, 2332, 3454, 4286, 2970, 1888, 1613, 2700, 3682, 4550, 3113, 1077, 1165, 890, 980, 573, 753, 654, 1659, 1296, 1010, 1313, 2357, 125, 693, 400, 999, 2345, 3452, 2765, 1987, 1828, and 1525 dollars. What were his total expenditures that year?

Art. 34.* Numbers expressed in other scales of notation are added on the same plan as decimal numbers, viz.: *carrying when the sum of any column is greater than the number of units in that order.*

Ex. 35. What is the sum of *three, two,* and *three,* expressed in the binery notation? Ans. 1000. (Eight.)

WRITTEN PROCESS.

Three is	1 1
Two is	1 0
Three is	1 1
Eight is 10	0 0

EXPLANATION.

The sum of the column of units is *two*, which, in the binary notation, is 1 *two* and 0, or 10. Write the 0 units under the units, and carry the 1 *two* to the column of *twos*, making its sum four twos, which equal 1 *eight*, 0 *fours*, 0 *twos*. Write a 0 under the column of twos, another under the fours, and 1 under the eights, thus making the answer 1 *eight*, 0 *fours*, 0 *twos*, 0 *units*.

*If desired, the learner can be exercised in the addition of numbers in any supposed notation, as explained in the preceding chapter. Or this may be omitted as a mere generalization of theoretical, but not practical interest or value; or it may be deferred till the pupil has learned to reduce numbers from the decimal to other scales.

CHAPTER IV.

SUBTRACTION OF SIMPLE WHOLE NUMBERS.

Art. 35. Subtraction is the process of finding the difference between two numbers.

The **terms** in subtraction are, *minuend*, *subtrahend*, and *difference*, or *remainder*.

The **minuend** is that number from which a number is taken.

The **subtrahend** is that number which is taken from the minuend.

The **difference**, or **remainder**, is the number which is left after subtraction.

Art. 36. The **sign** of subtraction is a short line placed in the line of writing, thus —. It is called *minus*. It indicates that the number following it is to be taken from the number before it. Thus, $8 - 5 = 3$ is read *eight minus five equals three*, or, *is equal to three.*

Art. 37. The minuend, subtrahend, and difference must be LIKE *numbers*.

NOTE.—In such expressions as "There are ten more horses than cows in the field," we merely consider these objects as far as they are of the same kind, that is, as *animals*.

Art. 38. The subtraction of numbers containing more than one order of units, is based upon the following principles:—

I. *The whole difference of two numbers is equal to the sum of the differences of their corresponding parts.*

II. *Units of any order in the subtrahend can be directly taken only from units of the same order in the minuend.*

III. *If both minuend and subtrahend be equally increased, the difference will not be changed.*

Ex. 1. From 735 take 349.

WRITTEN PROCESS.	EXPLANATION.
735 349 ——— 386	Since 9 units cannot be taken from 5 units, we add 10 units to 5 units, making 15 units; then 9 units from 15 units leave 6 units. Write 6 under the column of units. Since we *increased the minuend* by 10, we *increase the subtrahend* by 10, in order *not to alter the difference.* (Prin. III.) Therefore, we add 1 ten to the 4 tens of the subtrahend, making it 5 tens. Since 5 tens cannot be taken from 3 tens, we add 10 tens to the 3 tens, making 13 tens: then, 5 tens from 13 tens leave 8 tens. Write 8 under the column of tens. Since we increased the minuend by 10 tens, or 1 hundred, we increase the subtrahend as much by adding 1 hundred to the 3 hundreds, making 4 hundreds: then 4 hundreds from 7 hundreds, leave 3 hundreds, making the whole difference 386.

MENTAL AND RECITATION MODEL.

9 from 5 impossible; 9 from 15 leaves 6. Write 6, and carry 1 to 4, making 5: 5 from 3 impossible: 5 from 13 leaves 8. Write 8, and carry 1 to 3, making 4: 4 from 7 leaves 3. Write 3, making the whole remainder 386.

Rule. *Write the subtrahend under the minuend, units under units, tens under tens, &c., and draw a line under them.*

Beginning with units, subtract successively each figure from the figure above it, if possible, and write the difference below.

When a figure exceeds in value the figure above it, add 10 to the upper, from this sum subtract the lower, and add one to the next lower figure, and proceed with the result as before.

NOTE.—The process of adding 10 to the upper figure is often called *borrowing ten.* That of adding one to the next lower figure is called *carrying one.*

METHODS OF PROOF.

I. Add the remainder to the subtrahend; the result should equal the minuend.

II. Subtract the remainder from the minuend; the result should equal the subtrahend.

III. Add together the figures of the minuend, dropping every *nine* as fast as formed. Do the same with the subtrahend and remainder. The excess in the minuend should equal that in the subtrahend and remainder together.

IV. Beginning at the right, add the first, third, &c., that is, the alternate figures of the minuend, dropping *eleven* as fast as formed. Do the same with the second, fourth, &c., and subtract the last result from the former, increased by 11 if necessary.

Proceed in the same manner with the figures of the subtrahend and remainder. The last excess should equal that of the minuend.

ILLUSTRATION.

Minuend,	957384
Subtrahend,	425044
Remainder,	532340

In the minuend 4+3+5 = 12; drop 11, leaving excess 1; also 8+7 = 15; drop 11, leaving excess 4; 4+9 = 13; drop 11, leaving excess 2; from the excess of the first set of alternate figures, namely 1 (increased by 11, = 12,) take the excess of the second set, namely, 2, leaving 10, excess of minuend. In the subtrahend 4+0+2 = excess 6; also 4+5+4 = 13; drop 11, leaving excess 2; from the excess of the first set of alternate figures, namely, 6, take the excess of the second set, namely, 2, leaving 4, excess of subtrahend. In the remainder, 0+3+3 = 6, first excess; 4+2+5 = 11;—11 = 0, second excess; 6—0 = 6, excess of remainder; 4 (excess of subtrahend)+6, (excess of remainder) = 10, (excess of minuend).

	From	Take		From	Take
2.	40506	25364	7.	8765432	874443
3.	52739	6789	8.	7654321	359754
4.	81065	1687	9.	1222221	456789
5.	31415	23428	10.	1444443	456789
6.	14526	3547	11.	1006665	456789

12. How many are 864213579 — 24689753?
13. How many are 678901234 — 55444445?
14. How many are 567890123 — 3333334?
15. How many are 89012345 — 5555666?
16. How many are 890123456 — 7788899?

4

Find the difference

17. Between 90807060504030 and 10203040506070.
18. Between 1213141516171819 and 9181716151413121.
19. Between 3838373635343332 and 2333435363738393.
20. What number added to 123456789 makes ten billions?
21. What number taken from 5544332 leaves 445678?

Examples Combining Addition and Subtraction.

1. A farmer having 4120 sheep, sold to A 827, to B 608, and to C 979: how many had he remaining?

2. A had 367 sheep, B 139 more than A; and C had as many as A and B less 487: how many had C?

3. An army entered battle with 75253 men. Of these 3488 were killed, 8564 were wounded, 4739 were taken prisoners, and 1367 deserted. How many remained in the ranks?

4. The distance from Pittsburgh to Philadelphia is 355 miles. When one locomotive is 188 miles from Pittsburgh, and another 109 miles from Philadelphia, how far are they apart? How far apart, when one has traveled 197 miles, and the other 216 miles?

The symbol $ denotes dollars.

5. A had in bank $1603; he drew at one time $538, at another $307, at another $640; how much remained?

6. On a debt of $3750 the following payments were made:—$525, $475, $225, $155, $635, and $150; how much remained due, not reckoning interest?

7. A man had a sum of money; afterward he earned $4590, then spent $784, and had remaining $8203; how much had he at first?

8. A man dying, left $70000, to be divided among his widow, two sons, and three daughters; each son received $10800, each daughter $7250, and the widow the remainder. What was the widow's share?

Art. 39. Numbers expressed in other scales of notation than the decimal are subtracted on a similar plan.

Ex. 1. From *twenty-five*, expressed in the quinary notation take *eighteen*. Ans. Seven. (12, equal to 1 *five* 2 *units*.)

WRITTEN PROCESS.				
		(25's)	(5's)	(1's)
Twenty-five	is	1	0	0
Eighteen	is		3	3
Seven	is		1	2

EXPLANATION.

Three from 0, impossible; 3 from 5 leaves 2. Write 2, and carry 1 to 3, making 4: 4 from 0, impossible: 4 from 5 leaves 1. Write 1, and carry 1 to 0, making 1: 1 from 1 leaves 0. Hence the remainder is 1 *five* 2 *units*, or seven.

EXAMPLES FOR PRACTICE.

1. From *forty-one* take *twenty-three*, expressed in senary notation.

(SUGGESTION. 65—35=30, or eighteen.)

2. From *one hundred* take *seventy-five* in octary notation.

(SUGGESTION. 144—113=31, or twenty-five.)

3. From *two hundred* take *eighty-seven* in nonary notation.

(SUGGESTION. 242—96=135,=one hundred thirteen.)

CHAPTER V.

MULTIPLICATION OF SIMPLE WHOLE NUMBERS.

Art. 40. Multiplication is the process of taking one number as many times as there are units in another.

Multiplication is, also, a short method of performing addition when the numbers to be added are equal.

The **terms** of multiplication are, *multiplicand*, *multiplier*, and *product*.

The **multiplicand** is the number to be taken.

The **multiplier** is the number which shows how many times the multiplicand is to be taken.

The **product** is the result obtained by multiplication.

The multiplicand and multiplier are called the **factors** of the product, because they make or produce it.

Art. 41. The product expresses the same kind of quantity as the multiplicand, for the *nature* of a quantity is not changed by repeating it.

The multiplier is an *abstract* number, because it merely shows the number of times the multiplicand is to be taken. It is often a concrete number in the conditions of a question, but must be regarded as abstract in multiplying. Thus, in solving the question "At 4 dollars a piece, how much do 5 hats cost?" we convert the concrete number *five hats* into an abstract number expressing *times*, thus: At 4 dollars a piece, 5 hats cost 5 *times* 4 dollars, which is 20 dollars.

Art. 42. The **sign** of multiplication is an oblique cross; thus, $\times$. It indicates that the number preceding it and the number following it are *factors*, but does not indicate which is the multiplier or multiplicand. It is read *multiplied by*, *times*, or *into*.

The product of two abstract numbers is the same, whichever is the multiplier. Thus, 5 times 7 = 7 times 5.

A **parenthesis**, (—), inclosing an expression of two or more terms, signifies that the value of that expression is treated as a single number. A line, called a **vinculum**, drawn above such an expression, signifies the same thing.

Thus, the expression $(5-3)\times 6$, and $\overline{5-3}\times 6$, both signify that the difference of 5 and 3 is to be multiplied by 6. The parenthesis and vinculum are sometimes called *signs of aggregation.*

Art. 43. When the factors are simple whole numbers of more than one figure, the process of multiplying is based on the following principles:—

I. *In abstract numbers, the product of any two factors is the same, whichever factor is used as the multiplier.*

II. *If the figure of any order be multiplied by units, the product is of the same order as that multiplied.*

III. *If the figure of any order be multiplied by tens, the product will be one order higher than it would be, if the figure were multiplied by units; if by hundreds, two orders higher; if by thousands, three orders higher; &c.*

IV. *The whole product is equal to the sum of the products of every part of the multiplicand by every part of the multiplier.*

Ex. 1. Multiply 246 by 8. Ans. 1968.

WRITTEN PROCESS.

Multiplicand,	2 4 6
Multiplier,	8
Product,	1 9 6 8

EXPLANATION.

Write the numbers so that units of the same order stand in the same column; then proceed thus: 8 times 6 units are 48 units = 4 tens and 8 units, write the 8 in the units' place, and add the 4 tens to the product of tens. Eight times 4 tens are 32 tens, + 4 tens are 36 tens = 3 hundreds and 6 tens, write the 6 tens in tens' place, and add the 3 hundreds to the product of hundreds. Eight times 2 hundreds are 16 hundreds, + 3 hundreds are 19 hundreds = 1 thousand and 9 hundreds, write 9 in the hundreds' place, and 1 in the thousands' place, making the whole product 1968.

Ex. 2. Multiply 5067 by 2384. Ans. 12079728.

WRITTEN PROCESS.

Multiplicand		5067	
Multiplier		2384	
Product by 4 units,	=	20268	units.
" " 8 tens,	=	40536	tens.
" " 3 hundreds,	=	15201	hundreds.
" " 2 thousands,	=	10134	thousands.
Total product,		12079728	

EXPLANATION.

Four times 5067 are 20268. Eight tens times 5067 are 40536 *tens.* Three hundred times 5067 are 15201 *hundreds.* Two thousand times 5067 are 10134 *thousands.* Hence, the sum of these products must be the sum of 2 thousands' 3 hundreds' 8 tens' 4 units' times 5067, or 2384 times 5067. Hence, adding the products of the parts, called *partial products*, we obtain the *whole product*, 12079728.

Rule. *Write the multiplier under the multiplicand, units under units, tens under tens, &c., and draw a line under them.*

Beginning with the units, multiply successively each figure of the multiplicand by each significant figure of the multiplier, writing the right figure of each product, and carrying the left, as in addition.

When the multiplier has more than one significant figure, write the first figure of each partial product under that figure of the multiplier which produced it, and the other figures so as to form columns with the figures in the lines above it.

Add the partial products; their sum is the whole product.

METHODS OF PROOF.

I. Multiply the multiplier by the multiplicand. The results should be equal.

II. Add together the figures of the multiplicand, dropping from the reckoning every *nine* as fast as formed. Do the same with the multiplier, and with the product. Multiply the last *xcess over nine in* the multiplicand by that in the multiplier,

and reject the nines from *this* product. The excess over nine in this product should equal that in the general product.

Ex. 3. Multiply 935 by 546, and prove the work by rejecting the nines. Ans. 510510.

WRITTEN PROCESS.

935	Excess 8	
546	" 6	
5610		48, Excess 3
3740		
4675		
510510		Excess 3

EXPLANATION.

In the multiplicand drop 9, and the other figures, 3+5=8, excess over 9.

In the multiplier 5+4=9; drop it, and 6 is the excess. The product of these excesses is 6×8=48; 4+8=12; drop 9, and 3 is the excess.

In the product, 5+1+5+1 =12: drop 9, and 3 is the excess; the same as the excess in the product of the excesses.

III. Beginning at the right, add the alternate figures of the multiplicand, dropping every *eleven* as fast as formed. Do the same with the other figures, and subtract the last result from the former, increased by 11, if necessary. Proceed in the same manner with the multiplier. Multiply the excess in the multiplicand and that in the multiplier together, and drop the *elevens* in like manner from this product, and from the general product. The excesses in the two products should be equal.

Ex. 4. Multiply 9264397 by 9584, and prove the work by rejecting the elevens. Ans. 88789980848.

M'd 9264397	Excesses.	M'r 9584	Excesses.
9+6+3+7	= 3	5+4	= 9
2+4+9	= 4	9+8	= 6
Last excess, 4 from 3(+11)	= 10	Last excess, 6 from 9	= 3

Product of the two excesses = 10 × 3 = 30.

3 from 0(+11) = 8, excess in product of excesses.

Product 88789980848	Excesses.
8+7+9+8+8+8	= 4
8+8+9+0+4	= 7

Last excess, 7, from 4(+11) = 8, the same as excess in product of excesses in multiplier and multiplicand.

NOTE.—Some of the specific directions of the rule are merely matters of custom and convenience. All that is really necessary is to determine correctly the order of each figure in the partial products, and to find the sum of the products. Hence, *we may begin with any figure of the multiplier, and proceed with the other figures in any order*, provided we write the partial products so that they can be added.

EXAMPLES FOR PRACTICE.

5. Multiply 4607 by 706.
6. Multiply 45013 by 5403.
7. Multiply 24357 by 5009.
8. Multiply 253647 into 1468.
9. Multiply 556688 into 7806.
10. Multiply together 123, 456, and 789.
11. Multiply together 204, 305, and 607.
12. Multiply together 18, 27, 36, and 45.
13. $101 \times 203 \times 405 \times 79 =$ how many?
14. At the rate of 47 miles a day, how far would a man travel in 35 days? Ans. 1645 miles.

ANALYSIS.—If a man travel 47 miles each day, in 35 days he will travel 35 times 47 miles, that is, 1645 miles.

15. If a railroad 698 miles long costs $46089 per mile, what is the total cost?
16. At $53871 per day, what are the expenses of an army for 365 days?
17. What cost 45 pieces of broadcloth, containing 37 yards each, at 8 dollars per yard?
18. A drover bought 498 cattle at $56 each; what did he pay for them all?
19. If 15 men build a wall in 45 days, how long would it take 1 man to build it?
20. What will be the value of 304 horses, at $86 each?
21. How many pounds of flour are there in 387 barrels, there being 196 pounds in each barrel?
22. If 187 men can grade a street in 187 days, how long would it take 1 man to grade it?
23. If a regiment of soldiers consists of 1010 men, how many men are there in an army of 405 regiments?

24. There are 5280 feet in 1 mile, how many feet are there in 75 miles?

25. What is the value of 175 shares of oil stock at $125 per share?

26. If a furnace consume 367 bushels of coal in 1 day, how many bushels will it consume in 365 days?

POWERS OF NUMBERS.

Art. 44. A **power** of a number is the number itself, or the result obtained by using it *two or more* times as a factor.

An **exponent,** or **index,** is a small figure written to the right, and a little above a number to show how many times it is used as a factor.

The **first power** of a number is the number itself: thus, $5^1=5$.

The **second power,** or **square,** of a number is the result obtained by using it twice as a factor; thus, $5^2=5\times5=25$.

The **third power,** or **cube,** of a number is the result obtained by using it three times as a factor; thus, $5^3=5\times5\times5=125$.

The other powers are named in the order of their numbers; as *fourth power, fifth power, sixth power,* &c.

1. What is the square of 25? Ans. 625.
2. What is the fourth power of 23?
3. What is the cube of 45?
4. What is the fifth power of 12?
5. What is the seventh power of 6?
6. What is the sixth power of 7?
7. Multiply 16^2 by 5^3.
8. What is the product of 18^2 by 3^4?
9. Find the square, also the cube of all numbers less than 20.

Art. 45. A **composite number** is a number produced by integral factors, each greater than unity. Thus, 24 is a composite number, because it can be produced by 2×12, or 3×8, or 6×4, or $2\times3\times4$, or $2\times2\times6$, or $2\times2\times2\times3$.

The **component factors** of a number are such factors as, multiplied together, produce that number.

NOTE.—Care should be taken to distinguish between a *factor* of a number and a *part* of it. Factors, *multiplied*, produce the number. Parts, added, make the number. Thus, 5 and 4 are *parts* of 9, but not component *factors* of it.

SPECIAL METHODS OF MULTIPLICATION.

Art. 46. When the multiplier is 1 with a cipher or ciphers annexed; that is, some power of 10.

Ex. 1. Multiply 357 by 10. Ans. 3570.

To multiply 357 by 10 is to find 357 *tens*, which is done by writing 357 so as to end in the tens place, and filling the units place with a cipher. Hence, annexing 1 cipher to a number multiplies the number by 10. In the same way it can be shown that annexing two ciphers to a number multiplies that number by 100; annexing three ciphers, by 1000, &c.

Rule. *Annex to the multiplicand as many ciphers as there are in the multiplier.*

EXAMPLES FOR PRACTICE.

2. Multiply 64 by 10; by 100; by 1000; by 10000.
3. Multiply 402 by 10; by 100; by 1000; by 10000.
4. Multiply 930 by 10; by 100; by 100000.
5. Multiply 7500 by 100; by 10000; by 1000000.

CASE I.

Art. 47. When the multiplier is a composite number.

Ex. 1. What cost 63 acres of land at $256 per acre?

Ans. $16128.

SPECIAL PROCESS.

$$
\begin{array}{r l}
\$256 & \\
9 & \\
\hline
\$2304 & = 9 \times \$256 \\
7 & \\
\hline
\$16128 & = 7 \times 9 (\text{or } 63) \times 256
\end{array}
$$

EXPLANATION.

The factors of 63 are 7 and 9. At $256 an acre, 63 acres of land will cost 7 times 9 times $256: 9 times $256 are $2304, and 7 times $2304 are $16128.

Rule. *Resolve the multiplier into a convenient set of component factors. Multiply the multiplicand by one of these factors,*

the product thus obtained by another, and so on, till every factor has been used as a multiplier. The last product will be the product required.

EXAMPLES FOR PRACTICE.

2. What cost 72 tons of steel at $354 a ton?
3. What cost 56 wagons at $247 a piece?
4. What cost 84 mules at $96 a piece?
5. What cost 132 acres of land at $132 an acre?
6. Find the product of 7562 by 182 by two multiplications and no additions.
7. Find the product of 3579 by 224 by two multiplications and no additions.
8. Find the product of 478 by 323 by two multiplications and no additions.

CASE II.

Art. 48. When there are ciphers at the right of one, or both factors.

Ex. 1. Multiply 3600 by 240. Ans. 864000.

WRITTEN PROCESS.

```
 3600
 240
 ----
144
72
------
864000
```

EXPLANATION.

The factors of 3600 are 36 and 100, and the factors of 240 are 24 and 10. Since the product of these factors is the same in whatever order they are taken, it is plain that 240 times 3600 must be equal to 24 times 36 times 100 times 10. Now 24 times 36 is 864: 864 times 100 is 86400; 86400 times 10 is 864000.

Rule. *Multiply as if there were no right-hand ciphers, and annex to this product as many ciphers as are on the right-hand of both factors.*

EXAMPLES FOR PRACTICE.

2. Multiply 720 by 1400; by 3500; by 78.
3. Multiply 83600 by 25000; by 420000.
4. If it is 3500 miles from New York to Oporto, how many miles would a ship sail in making 20 complete trips?

Ans. 140000 miles.

5. Find the product of $20 \times 300 \times 4000 \times 50000$.

Art. 49. When one part of the multiplier is a factor of another part.

Ex. 1. Multiply 7264 by 486. Ans. 3530304.

SPECIAL PROCESS.	
7264	
486	
43584	Prod. by 6 units.
348672	Prod. by 48 tens.
3530304	

EXPLANATION.

The first figure, 6, of the multiplier is a factor of the next two figures, 48. Since 6 times the multiplicand is 43584, then 48 times the multiplicand must be 8 times 43584, or 348672; but, since the 48 is *tens*, the 348672 must be tens. Hence its right-hand figure is put in tens' place. The sum of these partial products equals the whole product of the multiplicand by each figure of the multiplier.

Rule. *First multiply by that part of the multiplier which is a factor of the other part: then find as many times that partial product as the other part is of the part used. Proceed in this way till all of the multiplier has been used. Place the right-hand figure of each partial product under the right-hand figure of that part of the multiplier which produced it, and add the partial products.*

(2.)

75823	
28735	
530761	Product by 7 hundreds.
2123044	Product by 28 thousands.
2653805	Product by 35 units.
2178773905	

(3.)

385265	
241686	
2311590	Product by 6 units.
9246360	Product by 24 ten thousands.
64724520	Product by 168 tens.
93113156790	

EXAMPLES FOR PRACTICE.

4. Multiply 70605 by 255: by 246: by 427: by 568.
5. Multiply 59374 by 13211: by 12111: by 10812: by 968.
6. Multiply 61728 by 819: by 963: by 369: by 639: by 847.
7. Multiply 43028 by 1684: by 4816: by 8416: by 1648.
8. Multiply 695872 by 729819 by three partial products.
9. Multiply 2030405 by 504567 by three partial products.
10. Multiply 869574 by 999891 by three partial products.

Art. 50. A large number of special forms of shortening multiplication can be developed under the following plans:—

I. When the multiplier is by some convenient quantity, called the **complement,** *less* than a power of 10, *annex to the multiplicand as many ciphers as there are in the given power, and subtract from it the product of the multiplicand by the complement.*

II. When the multiplier is by some convenient quantity, called the **excess,** *greater* than a power of 10, *annex to the multiplicand as many ciphers as there are in the given power, and add it to the product of the multiplicand by the excess.*

ILLUSTRATIONS.

1. $99\times567=100\times567$, less once $567=56700-567$.
2. $999\times567=1000\times567$, less once $567000-567$.
3. $98\times567=100\times567$, less twice $567=56700-1134$.
4. $990\times567=567000-(10\times567)$.
5. $9980\times567=5670000-(20\times567)$.
6. $101\times567=56700+567$.
7. $1001\times567=567000+567$.
8. $102\times567=56700+(2\times567)$.
9. $1002\times567=567000+(2\times567)$.
10. Find the product of 3589 by 9999, by a single subtraction.
11. Find the product of 721456 by 9970 by a *complement.*
12. Find the product of 357902 by 10090 by an *excess.*
13. Find the product of 849573 by 10009 by an *excess.*
14. Multiply 35276 by 995 by a *complement.*

15. Multiply 76845 by 9996 by a complement.
16. Multiply 123456789 by 9999999 by a complement.
17. Multiply 987654321 by 99999998 by a complement.
18. Multiply 695783064 by 9999997 by a complement.
19. Multiply 537486 by 1000800 by an excess.

Art. 51. To square a given number.

The product of the sum and difference of two numbers is equal to the difference of their squares.

ALGEBRAIC DEMONSTRATION.	NUMERICAL ILLUSTRATION.
$a + n$	$6 + 3 = 9$
$a - n$	$6 - 3 = 3$
$- an - n^2$	$-18 - 9$
$a^2 + an$	$36 + 18$
$a^2 \quad - n^2$	$36 \quad - 9 = 27$

REMARK.

It is a principal in Algebra that the product of a number by a subtractive or *minus* multiplier is subtractive or *minus.*

Art. 52. From the above demonstration and illustration it appears that *the square of a number is equal to the product of its sum after a certain increase by its difference after an equal decrease, plus the square of the added or subtracted number.* Thus, the square of 6, which is 36, is equal to the product of $6+3$, or 9, by $6-3$, or 3, (which product is 27,) plus the square of 3: that is, $36=27+9$.

EXERCISES.

1. $25\times25=30\times20+25=625.$
2. $45\times45=50\times40+25=2025.$
3. Find in the same way, 55^2: 65^2: 75^2: 85^2: 95^2: 35^2.
4. $36^2=32\times40+16$, or $42\times30+36=1296.$
5. Find in the same manner, 27^2: 48^2: 54^2: 63^2: 72^2: 81^2: 19^2: 18^2: 17^2: 16^2: 15^2: 14^2: 13^2: 23^2: 24^2: 26^2: 28^2: 29^2.

Art. 53. From Arts. 51 and 52, it appears that, if the less of two numbers be increased, and the greater diminished, by half their difference, their product then will be greater than before, by the square of half their difference.—Thus:

$$35\times25=30\times30-25=875.$$

EXERCISES.

1. $45\times35=40\times40-25=1575$.

2. Find in the same way, 25×35: 45×55: 55×65: 65×75: 75×85: 85×95.

3. $18\times22=20\times20-4=396$.

4. Find in the same way, 28×32: 37×43: 49×51: 56×64: 84×96: 93×107.

5. Find in the same way, 91×109: 92×108: 94×106: 95×105: 96×104: 97×103: 98×102: 99×101.

6. Find in the same way, 89×111: 88×112: 109×111: 108×112: 107×113: 106×114: 119×121: 117×123.

7. 115×125: 114×126: 113×127: 112×128: 111×129: 999×1001: 998×1002: 995×1005: 994×1006.

Art. 54.* Numbers expressed in other scales of notation than the decimal are multiplied on a similar plan.

Ex. 1. Multiply *forty-one* by *twenty-three* in senary notation.

Ans. 4211, senary; = 943, decimal.

WRITTEN PROCESS.

		(6's)	(1's)
Forty-one is		6	5
Twenty-three is		3	5
	5	4	1
3	2	3	
4	2	1	1

EXPLANATION.

Five times 5 units are 25 units, equal to 4 sixes, and 1 remains. Write 1, and carry 4. Five times 6 sixes are 30 sixes, and 4 carried make 34 sixes, equal to five 36's and 4 remain. Write 4 and carry 5. Five times no 36's are no 36's and 5 carried make five 36's; etc. In this we *think* decimally, and reduce results to senary notation.

Ans. 4 2 1 1 $= (1)+(1\times6=6)+(2\times36=72)+(4\times216=864)=943$.

EXAMPLES COMBINING ADDITION, SUBTRACTION, AND MULTIPLICATION.

1. A man bought two farms, one containing 168 acres at \$25 per acre, and the other 225 acres at \$48 per acre; what was the cost of both?

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*This Article may be omitted, if desired.

2. A bought a farm containing 173 acres at $43 per acre, B bought 165 acres at $37 per acre; how much more did A's farm cost than B's?

3. A has 349 sheep, B has three times as many — 708, and C has as many as A and B together; how many sheep has B and C each, and how many have they all?

4. Two persons start from the same point, and travel in opposite directions; one travels 28 miles a day, and the other 35 miles. How far apart will they be in 19 days?

5. A drover bought 135 head of cattle at $27 a head, and 78 head at $43 a head, and sold the whole lot at $35 a head; what was his entire profit or loss?

6. A man's yearly income is $2375; he pays for house-rent $437, his other expenses amount to 4 times as much — $205; how much does he save yearly?

7. Multiply $753-(84+46)$ by $(3^2\times 43)-(304-96)$.

8. A is worth $5250, B is worth $375 less than A, and C is worth as much as A and B together — $3107; how much is C worth?

9. If a cow cost $35, a horse 5 times as much, and a farm 10 times as much as the cow and horse together, — $309; how much more will the farm cost than 9 cows and 7 horses, at the same rate?

10. In an army of 17085 men, there were 1673 men killed in battle, and 4 times as many wounded, less the number that were taken prisoners, which was 1941; how many men remained in the army, and how many were wounded?

11. I exchanged a house worth $2160 and a store worth $2875, for 160 acres of land worth $43 per acre; how much do I still owe?

12. $(35^2+55^2-65^2)\times(33^2-32^2)=?$

CHAPTER VI.

DIVISION OF SIMPLE WHOLE NUMBERS.

Art. 55. Division is the process of finding how many times one number contains another, or of finding one of the equal parts of a number.

The **terms** of division are, *dividend, divisor, quotient,* and *remainder.*

The **dividend** is the number to be divided.

The **divisor** is the number by which we divide.

The **quotient** is the number which shows how many times the divisor is contained in the dividend.

The **remainder** is the excess of the dividend over as many times the divisor as there are whole units in the quotient. Thus, 3 is contained in 14 four times and 2 remain. In this case, because the remainder 2 is two-thirds of the divisor 3, the true quotient is *four* and *two-thirds.* The remainder expresses the same kind of quantity as the dividend, because it is a part of the dividend.

While dividing, we must consider divisor and dividend as *like numbers,* because the divisor is, for the time being, considered as one of the parts of the dividend. Also, the quotient must, in dividing, be considered an abstract number, because it merely shows the number of times the dividend contains the divisor.

ILLUSTRATION.—If 3 persons share equally $15, each receives as many dollars as 3 is contained times in 15. Since 3 is contained in 15 five times, each person receives $5. In this example 3 *persons* and 15 *dollars* are not like numbers, but, in dividing, we make them like numbers by making both abstract. Also, the quotient, 5 *times,* is abstract and we reason from it to the concrete number, 5 *dollars.*

Art. 56. As multiplication can be illustrated by *addition,* so division can be illustrated by *subtraction.* Thus, we can illustrate the fact that 27 contains 9 three times, by subtracting 9 successively from 27 till it is exhausted. We find that three subtractions of 9 exhaust 27.

Art. 57. The **signs of division** are ÷, —, and). Thus, $48 \div 6 = 8$, $\frac{48}{6} = 8$, $6)48 = 8$, denotes that 48 is to be divided by 6, and that the quotient is 8; and is read 48 *divided by* 6 *equals* 8.

Art. 58. The equal parts into which a quantity may be divided have names which correspond with their number.

A *half* of a quantity is one of two equal parts which make that quantity. Hence a quantity is composed of 2 halves of itself.

A *third* of a quantity is one of three equal parts which make that quantity. Hence a quantity is equal to 3 thirds of itself.

A *fourth* of a quantity is one of four equal parts which make that quantity. Hence a quantity is equal to 4 fourths of itself.

In this manner other parts are named, thus:—*fifths, sixths, sevenths, eighths, ninths, tenths, elevenths, twelfths, thirteenths,* &c.; *twentieths, twenty-firsts, twenty-seconds, twenty-thirds,* &c.; *one hundredths, one-hundred-and-firsts, one-hundred-and-seconds,* &c.

Art. 59. When the dividend is written over the divisor, the number thus formed may express four things, viz.:—

1st. *The division of the dividend into as many equal parts as there are units in the divisor.* Thus, $\frac{15}{3}$ may express the division of 15 into 3 equal parts.

2d. *The division of the dividend into equal parts, each of which has the same value as the divisor.* Thus, $\frac{15}{3}$ may express the division of 15 into *threes.*

3d. *One of those equal parts of the dividend whose name is represented by the divisor.* Thus, $\frac{15}{3}$ may express *one-third of 15.*

4TH. *As many of those equal parts of a unit whose name is represented by the divisor, as there are units in the dividend.* Thus, $\frac{15}{3}$ may express *fifteen thirds of* 1.

In the last case the expression is a *fraction,* because it expresses a certain number of the equal parts of a unit.

Art. 60. The number below the line of a fraction is called the **denominator,** because it *denominates,* or names the parts into which the unit is divided.

The number above the line of a fraction is called the **numerator,** because it *numerates,* or states the number of the parts expressed by the fraction

Art. 61. To read a fraction, *read first the number in the numerator, then the parts of a unit indicated by the denominator.* Thus, $\frac{2}{3}$ is read *two-thirds.*

EXERCISES.

1. Read $\frac{3}{4} : \frac{2}{5} : \frac{1}{6} : \frac{4}{7} : \frac{5}{8} : \frac{2}{9} : \frac{7}{10} : \frac{8}{11} : \frac{11}{12} : \frac{9}{13} : \frac{11}{14} : \frac{8}{15} : \frac{13}{16} : \frac{15}{17}$.
2. Read $\frac{19}{20} : \frac{18}{21} : \frac{15}{22} : \frac{6}{23} : \frac{11}{34} : \frac{13}{45} : \frac{39}{56} : \frac{41}{67} : \frac{53}{78}$.
3. Read $\frac{17}{100} : \frac{74}{101} : \frac{86}{102} : \frac{95}{103} : \frac{99}{204} : \frac{123}{305} : \frac{217}{406} : \frac{345}{507} : \frac{137}{625}$.

Art. 62. The method of dividing one number by another is based on the following principles:—

I. *The number of times that the whole dividend contains the divisor is equal to the number of times that all the parts of the dividend contain the divisor.*

II. *Any excess, occurring from one part of the dividend's more than containing the divisor, should be united with the rest of the dividend to form a new part, in order to find how many times it contains the divisor.*

III. *A final remainder, being smaller than the divisor, and not containing it, can only be represented as divided. Hence it should be written with the divisor under it in the form of a fraction.*

Ex. 1. Divide 789 by 5. Ans. $157\frac{4}{5}$.

WRITTEN PROCESS.	EXPLANATION.
Divisor. Dividend. 5)789 Quotient. |157$\frac{4}{5}$	5 is contained in 7 hundreds 1 hundred times, with a remainder 2 hundred. Unite the 2 hundreds with the 8 tens, making 28 tens; 5 is contained in 28 tens 5 *tens* times, with a remainder of 3 tens.

Unite the 3 tens with the 9 units, making 39 units · 5 is contained in 39 units 7 times, with a remainder 4. Since 4 does not contain 5, it must be *represented* as divided into five equal parts, each part being $\frac{1}{5}$ of 4, or $\frac{4}{5}$. Hence the complete quotient is 157$\frac{4}{5}$.

Ex. 2. How many times is 43 contained in 817254?

Ans. 19005$\frac{39}{43}$.

WRITTEN PROCESS.	EXPLANATION.
$$\begin{array}{l} 43)817254\,\lvert\,19005\tfrac{39}{43} \\ \underline{43} \\ 387 \\ \underline{387} \\ \quad 254 \\ \quad \underline{215} \\ \quad\; 39 \end{array}$$	If we do not know how many times 43 is contained in 81, we must *make a trial* with the left-hand figure, 4, of 43, and the left-hand figure, 8, of 81. Thus:—4 is contained in 8 *twice;* but, on trying 2 times 43, we find it 86, which is more than 81. Hence we try *once* 43, putting one in the quotient, and then multiply the divisor by this quotient figure; once 43 is 43, which

subtracted from 81, leaves 38; to this remainder we annex or bring down 7, the next figure of the dividend, and thus form 387, the next partial dividend; 43 is contained in 387 nine times, and no remainder; we bring down 2 the next figure of the dividend and say "*43 in 2 no times,*" and put 0 in the quotient, and bring down the next figure 5. Then "*43 in 25 no times,*" and we put another 0 in the quotient, and bring down 4, making 254. Then, 43 in 254 5 times with a remainder 39. The remainder 39 is *represented* as divided, making the complete quotient 19005$\frac{39}{43}$.

NOTE.—When the division is performed mentally, and only the result written, the operation is called *Short Division;* but when all the work is written, it is called *Long Division.*

Rule. *Find how many times the divisor is contained in the fewest left-hand figures of the dividend that will contain it; this number of times is the first figure of the quotient.*

Multiply the divisor by this quotient figure, and subtract the product from that part of the dividend which was used.

To the right of the remainder, if there is any, annex the next figure of the dividend.

Find how many times the divisor is contained in the number thus formed; this number of times is the second figure of the quotient.

If there is no remainder after subtracting a product, find how many times the divisor is contained in the next figure of the dividend.

Proceed in this manner till every figure of the dividend has been used.

If there is a final remainder, place it, with the divisor under it, at the right-hand of the quotient.

Methods of Proof.

I. Multiply the divisor and quotient together, and to the product add the remainder if there is any; the result should equal the dividend.

NOTE.—In comparing division with multiplication, we perceive that the dividend is a *product*, of which the divisor and quotient are component factors.

II. Subtract the remainder, if there is any, from the dividend, and divide the result by the quotient: the new quotient should equal the first divisor.

III. Add together the figures of the divisor, dropping from the reckoning every *nine* as fast as formed. Do the same with the dividend, quotient, and remainder. Multiply the excess over *nines* in the divisor by that in the quotient; reject the nines from this product: to its excess add the excess of the remainder. The excess in the figures of this sum should equal that in the figures of the dividend.

IV. Beginning at the right-hand, add the alternate figures of the divisor, dropping every *eleven* as fast as formed. Do the same with the other figures, and subtract the last result from the former, increased by 11, if necessary. Proceed in the same manner with the quotient. Multiply these remainders together; to the product add the remainder, if any occurred in dividing; reject the *elevens* from this sum, in like manner; the excess should equal the excess of elevens in the figures of the dividend, found in the same manner.

Ex. 3. How many 13's are there in 189? Ans. 14. Rem. 7.

WRITTEN PROCESS.	PROOF SECOND.
13) 189	189
Quot. | 14 Rem. 7	7
	14) 182
New quot. = first divisor.	13
(See Proof Second.)	

NOTE.—This question does not specifically ask for the number of times that 13 is contained in 189, which is $14\frac{7}{13}$ times, but for the number of 13's in 189, which is fourteen 13's with an excess of 7. In proof, we take the excess from 189, and divide the rest into 14 equal parts, that is, find one-fourteenth of it, and find it to be 13.

4. How many 7's in 2579863? Quot. 368556. Rem. 6.
5. How many 9's in 898888888? Rem. 1.
6. How many 11's in 1077665544? Rem. 10.
7. How many 16's in 123000957? Rem. 13.
8. How many 17's in 37987654? Rem. 15.
9. How many 18's in 35884003? Rem. 11.
10. How many 19's in 234567689? Rem. 16.
11. How many 20's in 111975326108? Rem. 8.
12. How many 21's in 1888902990? Rem. 9.
13. How many 22's in 200091060? Rem. 4.
14. Gave $83698 for 67 acres of land. What was the cost per acre? Ans. $1249\frac{15}{67}$ dollars.

ANALYSIS.—The cost per acre was as many dollars as 67, the number of acres is contained times in 83698, the number of dollars of cost.

WRITTEN PROCESS.		PROOF THIRD, BY REJECTING NINES.	
67)83698($1249\frac{15}{67}$		Excess of nines in 67 =	4
67		" " " " 1249 =	7
166		Product of 4 by 7 =	28
134	bro't up.	Excess of nines in 28 =	1
	618	" " " " 15 =	6
329	603		
268		Sum of 1 and 6 =	7
	15	Excess in 83698 =	7
carried up.			

NOTE.—Prove this and the following examples by rejecting the nines.

15. Sold 74 acres of land for 15500 dollars. What was the price per acre?

16. A company of 95 soldiers cost the government 29400 dollars one year. What was the cost per man?

17. The wages of 203 men one year were 163400 dollars. How much would they be per man, if divided equally?

18. If 196 pounds of flour make a barrel, how many barrels are there in 14253647 pounds of flour?

19. At 275 miles per day, in how many days will a steamer travel 9876 miles?

20. At 5147 pieces of goods per day, in how many days will a factory make 1568900 pieces?

21. At 123456 dollars per day, in how many days would a debt of 555555555 dollars be paid?

22. Divide 555555555555 by 654321.

23. Divide 233687432712 by 930079.

Ans. $251255\frac{433567}{930079}$.

WRITTEN PROCESS.	PROOF FOURTH, BY REJECTING ELEVENS.	
930079)233687432712(251255		
1860158	9+0+3=12 : excess 1 : +11=	12
———	7+0+9=16 : " 5	5
4767163		—
4650395	Excess of 11's in divisor=	7
———	5+2+5=12 : excess 1 : +11=12	
1167682	5+1+2= 8 : 8	
930079	Excess of 11's in quotient	4
———		—
2376037	Product of excesses =	28
1860158	433595=433567+	28
———	3+5+5=13 : excess 2 : +11=	13
5158791	4+3+9=16 : " 5	5
4650395		—
———	Excess of 11's in sum=	8
5083962	Ex. of 11's in 1st set of div'd 6 : +11=	17
4650395	" 2d " 9	9
———		—
433567	Excess of 11's in div'd	8

Prove the following examples by rejecting 11's:—

24. Divide 51830567490 by 65027
25. Divide 62738495060 by 63845
26. Divide 791450038206 by 78901
27. Divide 802640135700 by 45273
28. Divide 937158004620 by 462087
29. Divide 162738495061 by 541672
30. Divide 2132435465768 by 678904
31. Divide 3458769210000 by 708095
32. Divide 4682010305003 by 830946
33. Divide 594061324342 by 9380746

34. If the expenses of a government are $514852 per day, in how many days would they amount to $47366384?

Ans. 92.

ANALYSIS.—At 514852 dollars per day, it would take as many days for the expenses to amount to 47366384 dollars as 514852 dollars is contained times in 47366384 dollars, that is, 92 days. In this problem our intention is to find how many parts of the given value 514852 dollars there are in the dividend 47366384 dollars.

35. If in 92 days the expenses of a government are 47366384 dollars, at what average rate is that per day?

Ans. 514852 dollars.

ANALYSIS.—If in 92 days the expenses are 47366384 dollars, the average rate per day is $\frac{1}{92}$ of 47366384 dollars, that is 514852 dollars. In this problem our intention is to separate the dividend into 92 equal parts, in order to find the value of each of those parts.

NOTE.—In solving the following examples give the *full analysis*, and *state the intention of the operator*.

36. At 258 dollars per acre, how many acres of land can be bought for 799542 dollars?

37. If 799542 dollars buy 3099 acres of land, what is the cost per acre?

38. C's income in 365 days was 148700 dollars. What was his income per day.

39. At 47 dollars a day, in how many days does an income amount to 14476 dollars?

40. At 24 miles an hour, in how many hours would a train go the distance around the globe, 24857 miles?

41. If a train moves uniformly 24857 miles in 43 days, what is the rate per day?

42. At 37 miles per day, in how many days would a person walk 3278 miles?

43. If a certain fixed star is 908070605040302010 miles from the earth, and if light moves 185736 miles per second, in how many seconds would the light of that star reach the earth?

44. At 31556929 seconds per solar year, in how many solar years would the light of that star reach the earth, neglecting fraction in dividend?

45. In 345 days the debt of a certain government, at war, increased 8997744444 dollars. At what rate per day did its debt increase?

46. A railroad, four hundred and seventy-eight miles long, cost, for building and equipments, eighteen millions seventy-five thousand and ninety-seven dollars. What was the total cost per mile?

47. Its earnings in 365 days were two millions five hundred thirty-eight dollars. What were the earnings per mile? Per day? Ans. Per mile, $4185\frac{108}{478}$. Per day, $5480\frac{338}{365}$.

48. If the treasury of the government contained, at a certain time, 99900000 dollars in five-dollar pieces, and a person were to count the money piece by piece, at the rate of 5 pieces per second, in how many seconds would he count the whole? Ans. 3996000.

49. If he worked 10 hours a day, 3600 seconds making an hour, in how many days would he count the whole? Ans. 111.

50. If the average hourly work of a compositor is 1400 em's, in how many days of 10 hours each would he set up a book of 480 pages, of 39 lines each, and 35 em's per line? Ans. 46 days 8 hours.

51. If the pump at the water-works of a city throws 123 gallons per stroke, and makes 18 strokes per minute, in how many hours of 60 minutes each, would it fill the reservoir, containing 5809375 gallons? Ans. $43\frac{97255}{132840}$ hours

Construction, Reading, and Verification of Formulas.

Art. 63. To *construct a formula* is to express by symbols the relations which quantities have in a problem. (Art. 29.)

To *read a formula* is to express in words the relations signified by the symbols of that formula.

To *verify a formula* is to prove by computation the correctness of its statements. (See Art. 42.)

Exercises.

Construct and verify the formula for

1. The sum of 23 and 45 is 68. Ans. $23+45=68$.
2. The difference of 38 and 17 is 21. Ans. $38-17=21$.
3. The product of 15 and 19 is 285. Ans. $15\times19=285$.
4. The quotient of 96 by 12 is 8.

   Ans. $96\div12=8$; or $\frac{96}{12}=8$.
5. The sum of 43 and 62, less the difference of 89 and 46 is 62. Ans. $(43+62)-(89-46)=62$.
6. The difference of the sum of 96 and 77, and the difference of 83 and 36 is 126. (Ans. $96+77)-(83-36)=126$.
7. The product of the quotient of 112 by 7, and of the sum of 56 and 64 by the difference of 20 and 35 is 128.

$$\text{Ans.}\quad \frac{112}{7}\times\frac{56+64}{35-20}=128;\ \text{or}\ \left(\frac{112}{7}\right)\times\left(\frac{56+64}{35-20}\right)=128;\ \text{or}$$

$$(112\div7)\times\Big((56+64)\div(35-20)\Big)=128.$$

Note.—It is not necessary to place the sign of multiplication between parentheses, or between a parenthesis and a number, to indicate their product. The absence of any sign between them also indicates multiplication. Thus, $(6+4)(4-2)=20$ indicates that the product of the sum of 6 and 4 by the difference of 4 and 2 is 20.

8. The quotient of the product of 25 and 45 by the sum of 12 and 13 equals the product of the quotient of the difference of 148 and 43 by the sum of 3 and 4 by the quotient of 63 by 21.

$$\text{Ans.}\quad \left(\frac{25\times45}{12+13}\right)=\left(\frac{148-43}{3+4}\right)\left(\frac{63}{21}\right).$$

9. The sum of the quotients of 175 by 7, and of 180 by 9, is equal to the difference of the quotients of 576 by 12, and of 117 by 3, multiplied by the difference of 14 and 9.

10. The difference of the differences of 231 and 123, and of 482 and 284, divided by the sum of the differences of 345 and 339, and of 199 and 203, is equal to the product of the products of 8 by 9, and of 18 by 3, less the sum 2164 and 1715.

11. The quotient of the product of the sum of 8 and 5 by the quotient of 48 by 6, divided by the difference of the quotients of 112 by 4, and of 72 by 36, is equal to the sum of the products of the quotients of 6 by 3, and of 10 by 5, and the differences of the quotients of 42 by 6, and of 56 by 8.

12. The sum of the quotient of 27 times 4 divided by the quotient of 6 times 9 by 18, and 12 times the sum of 18 and 11 times the difference of 13 and 6, is 1296.

13. The quotient of the quotient 98 by 7, divided by the quotient of 246 by 123, equals the sum of the sums of 17 and 19 and of 11 and 13, less the difference of the differences of 95 and 71 and of 169 and 92.

Verify the following formulas:—

14. $\left(\frac{51-16}{7}\right)\left(\frac{105\times 2}{10}\right)+(48\div\overline{3\times 8})-7=100.$

15. $(36\times 6)\div\left(\frac{8\times 54}{18}\right)+4\times\overline{16+6(8-5)}=145.$

16. $\left(\frac{4\times 429}{12}\right)\div\overline{7+8(23-6)}=\left(\frac{3(5-2(5-4))}{3+2(11-4(6-4))}\right)\left(\frac{7+5}{24\div 2}\right).$

17. $(15+12)(15-12)=(15\times 15)-(12\times 12)$. (See Art. 51.)

18. $(15+12)(15-12)+(12\times 12)=(15\times 15).$

19. $(15+12)+(15-12)=2\times 15.$

20. $(15+12)-(15-12)=2\times 12.$

21. $(20+3)(20+3)=(20\times 20)+2(20\times 3)+(3\times 3).$

22. $(20+5)(20+5)(20+5)=(20\times 20\times 20)+(3\times 5)(20\times 20)+(3\times 20)\times(5\times 5)+(5\times 5\times 5).$

Special Methods of Division and Multiplication.

Art. 64. To divide by 1 with a cipher, or ciphers, annexed, that is, by a power of 10.

Ex. 1. How many *tens* are there in 63975? Ans. $6397\frac{5}{10}$.

Analysis.—Since that part of 63975 which ends in the tens' place is 6397, and 5 units remain, which are $\frac{5}{10}$ of ten; there are $6397\frac{5}{10}$ *tens* in 63975.

Ex. 2. Find one-hundredth of 63975. Ans. $639\frac{75}{100}$.

Analysis.—To find one-hundredth of 63975 is to find how many *hundreds* there are in 63975. Since that part of 63975 which ends in the hundreds' place is 639, and 75 units remain, which are $\frac{75}{100}$ of a hundred, there are $639\frac{75}{100}$ hundreds in 63975; or $639\frac{75}{100}$ is one-hundredth of 63975.

Illustration First.	Illustration Second.
$1|0)6397|5=6397\frac{5}{10}$	$1|00)639|75=639\frac{75}{100}$

Note.—To show the advantage of this short process, let the learner divide 63975 by 10, and 100, by Long Division, and compare the two processes.

Rule. *Cut off from the right of the dividend as many figures as there are ciphers on the right of the divisor. The figures thus cut off are the remainder, and the other part of the dividend is the quotient.*

Examples for Practice.

	Divide.	Ans.		Divide.
3.	542 by 10.	$54\frac{2}{10}$.	12.	9067 by 1000.
4.	793 by 10.		13.	50023 by 10000.
5.	8125 by 100.	$81\frac{25}{100}$.	14.	70308 by 10000.
6.	2736 by 100.		15.	490258 by 10000.
7.	5793 by 1000.	$5\frac{793}{1000}$.	16.	8610372 by 100000.
8.	4862 by 1000.		17.	7501538 by 100000.
9.	3700 by 100.	37.	18.	6400721 by 100000.
10.	8500 by 100.		19.	3000469 by 1000000.
11.	8003 by 1000.	$8\frac{3}{1000}$	20.	5102034 by 1000000.

21. How many times is ten million contained in five hundred seventy-three million forty thousand and six?

Ans. $57\frac{3040006}{10000000}$.

Art. 65. To divide by a composite number, using its component factors.

CASE. I.

When there is no cipher on the right of the divisor.

Ex. 1. Divide 1908 by 36, using the factors 4 and 9.

Ans. 53.

PROCESS FIRST.	PROCESS SECOND.
4)1908 *units.*	9)1908 *units.*
9) 477 *fours.*	4) 212 *nines.*
53 *thirty-sixes.*	53 *thirty-sixes.*

EXPLANATION FIRST.—Dividing by 4, the quotient is 477 fours. Dividing these 477 fours into groups of 9 fours each, there are 53 such groups, that is, 53 groups of 36 each. Therefore, 53 is one-thirty-sixth of 1908.

EXPLANATION SECOND.—Dividing by 9, the quotient is 212 nines. Dividing these 212 nines into groups of 4 nines each, there are 53 such groups, that is, 53 groups of 36 each. Therefore, 53 is one-thirty-sixth of 1908.

Ex. 2. Divide 2719 by 72, using the component factors 3, 4, and 6.

Ans. $37\frac{55}{72}$.

WRITTEN PROCESS.

	Remainders.
3)2719 *units.*	
4) 906 *threes*, + 1 *unit* =	1
6) 226 *twelves*, + 2 *threes* =	6
37 *seventy-twos*, + 4 *twelves* =	48
Whole, or true remainder =	55

EXPLANATION.

Dividing by 3, the quotient is 906 *threes*, and 1 unit remains. Dividing these 906 threes into groups of 4 threes each, there are 226 such groups, that is, 226 *twelves*, and 2 *threes*, or 6 *units*, remain. Dividing these 226 twelves into groups of 6 twelves each, there are 37 such groups, that is, 37 *seventy-twos*, and 4 *twelves*, or 48 *units* remain. The complete remainder is the sum of these remainders, that is, 55 units. Because the whole divisor is 72, and the whole remainder 55, the complete quotient must be $37\frac{55}{72}$.

Rule. *Find any convenient set of component factors of the divisor, and divide the dividend first by one factor, then that quotient by another, and so on till every factor of the set has been used as a divisor. The last quotient will be the true quotient.*

Multiply each remainder by all the divisors preceding the one that produced it, and add together these products with the remainder produced by the first divisor. The sum will be the true remainder.

EXAMPLES FOR PRACTICE.

3. Divide 25861 by $105=3\times5\times7$. Ans. $246\frac{31}{105}$.
4. Divide 28450 by $77=7\times11$. Rem. 37.
5. Divide 1746485 by $504=7\times8\times9$. Rem. 125.
6. Divide 1053984 by $2310=2\times3\times5\times7\times11$. Rem. 624.
7. Divide 1910877 by $625=5\times5\times5\times5$. Rem. 252.
8. Divide 5495770 by $1287=9\times11\times13$. Rem. 280.
9. Divide 10650913 by $2592=4\times6\times9\times12$. Rem. 385.
10. Divide 12151568 by $4845=15\times17\times19$. Rem. 308.

CASE II.

When the right-hand part of the divisor is a cipher, or ciphers.

Ex. 1. Divide 57347 by 800. Ans. $71\frac{547}{800}$.

WRITTEN PROCESS.

$$\begin{array}{r|l} 8\,|\,00)\,573 & 47 \\ \hline 71 & 547 \end{array}$$

EXPLANATION.

One set of component factors of 800 is 100 and 8. Dividing by 100, there are 573 hundreds, and 47 units remain. Dividing 573 hundreds into groups of 8 hundreds each, there are 71 such groups, and 5 hundreds remain. Therefore the true remainder is $500+47=547$, and the complete quotient is $71\frac{547}{800}$.

Rule. *Cut off the cipher or ciphers from the right of the divisor, and as many figures from the right of the dividend. Then divide the remaining figures of the dividend by the remaining figures of the divisor. If dividing produces a remainder, annex to it the figures cut off from the dividend; the result is the true remainder. If dividing does not produce a remainder, the figures cut off from the dividend are the true remainder.*

EXAMPLES FOR PRACTICE.

Divide	Ans.	Divide
2. 639 by 40.	$15\frac{39}{40}$.	8. 107027 by 13000.
3. 8207 by 4100.	$2\frac{7}{4100}$.	9. 15536925 by 30700.
4. 98003 by 7000.	$14\frac{3}{7000}$.	10. 15536925 by 307000.
5. 83051 by 1600.	$51\frac{1451}{1600}$.	11. 24751637 by 40900.
6. 83051 by 16000.	$5\frac{3051}{16000}$.	12. 24751637 by 409000.
7. 107027 by 1300.	$82\frac{427}{1300}$.	13. 24751637 by 4090000.

Art. 66. To perform long division without writing the partial products.

Ex. 1. Divide 439752 by 48. Ans. $9161\frac{24}{48}$.

WRITTEN PROCESS.

```
48)439752(9161
     7
    29
      7
     24
```

EXPLANATION.

The divisor 48 is contained in 439 nine times. We now multiply 48 by 9, but instead of writing the product, we observe what figures, added to the product, would equal the corresponding figures of the dividend, and write them as the remainder. Thus, 9 times 8 are 72, and 7 make 79: write the 7 as the remainder: 9 times 4 are 36, and 7 to carry from 79 make 43. Since this equals the 43 of the dividend, there is no remainder except 7. Hence the new dividend is 77: the next remainder is 29: the next dividend is 295: the next remainder is 7: the last dividend is 72: the last remainder is 24: the whole quotient is $9161\frac{24}{48}$.

Rule. *Divide as in Long Division, subtracting each figure of a partial product as soon as it is obtained, and writing only the remainders and quotient figures.*

NOTE.—This is called the *Italian Method* of Long Division.

EXAMPLES FOR PRACTICE.

2. Divide 186575 by 37. Ans. $5042\frac{21}{37}$.
3. Divide 234610 by 15.
4. Divide 475381 by 27.
5. Divide 683502 by 13. Ans. $52577\frac{1}{13}$.
6. Divide 562083 by 75.

Art. 67. To contract multiplication by division.

CASE I.

When the multiplier is a convenient part of a power of 10.

Ex. 1. Multiply 3857 by 25. Ans. 96425.

WRITTEN PROCESS.

$$\begin{array}{r} 4\,)\,3\,8\,5\,7\,0\,0 \\ \hline 9\,6\,4\,2\,5 \end{array}$$

EXPLANATION.

Twenty-five times 3857 is one-fourth of 100 times 3857, because 25 is one-fourth of 100. Hence, if we annex two 0's to 3857, thus multiplying it by 100, and divide the result by 4, we obtain 25 times 3857.

Rule. *Annex to the multiplicand as many ciphers as there are in 100,* or *1000, &c., and find that part of the result which the multiplier is of 100, or 1000, &c.*

To give to this rule the greatest utility, it is necessary that the computer should develop and remember as many facts as possible like those in the following

TABLE OF PARTS.

Of 100		Of 1000	
$6\frac{1}{4} = \frac{1}{16}$	$56\frac{1}{4} = \frac{9}{16}$	$62\frac{1}{2} = \frac{1}{16}$	$500 = \frac{1}{2}$
$8\frac{1}{3} = \frac{1}{12}$	$62\frac{1}{2} = \frac{5}{8}$	$83\frac{1}{3} = \frac{1}{12}$	$562\frac{1}{2} = \frac{9}{16}$
$9\frac{1}{11} = \frac{1}{11}$	$68\frac{3}{4} = \frac{11}{16}$	$90\frac{10}{11} = \frac{1}{11}$	$625 = \frac{5}{8}$
$11\frac{1}{9} = \frac{1}{9}$	$75 = \frac{3}{4}$	$111\frac{1}{9} = \frac{1}{9}$	$666\frac{2}{3} = \frac{2}{3}$
$12\frac{1}{2} = \frac{1}{8}$	$83\frac{1}{3} = \frac{5}{6}$	$125 = \frac{1}{8}$	$750 = \frac{3}{4}$
$14\frac{2}{7} = \frac{1}{7}$	$87\frac{1}{2} = \frac{7}{8}$	$142\frac{6}{7} = \frac{1}{7}$	$833\frac{1}{3} = \frac{5}{6}$
$16\frac{2}{3} = \frac{1}{6}$	$112\frac{1}{2} = 1\frac{1}{8}$	$166\frac{2}{3} = \frac{1}{6}$	$875 = \frac{7}{8}$
$18\frac{3}{4} = \frac{3}{16}$	$114\frac{2}{7} = 1\frac{1}{7}$	$187\frac{1}{2} = \frac{3}{16}$	$937\frac{1}{2} = \frac{15}{16}$
$25 = \frac{1}{4}$	$116\frac{2}{3} = 1\frac{1}{6}$	$250 = \frac{1}{4}$	$1125 = 1\frac{1}{8}$
$31\frac{1}{4} = \frac{5}{16}$	$120 = 1\frac{1}{5}$	$312\frac{1}{2} = \frac{5}{16}$	$1142\frac{6}{7} = 1\frac{1}{7}$
$33\frac{1}{3} = \frac{1}{3}$	$125 = 1\frac{1}{4}$	$333\frac{1}{3} = \frac{1}{3}$	$1166\frac{2}{3} = 1\frac{1}{6}$
$37\frac{1}{2} = \frac{3}{8}$	$133\frac{1}{3} = 1\frac{1}{3}$	$375 = \frac{3}{8}$	$1250 = 1\frac{1}{4}$
$50 = \frac{1}{2}$	$150 = 1\frac{1}{2}$	$437\frac{1}{2} = \frac{7}{16}$	$1333\frac{1}{3} = 1\frac{1}{3}$

EXAMPLES FOR PRACTICE.

2. Multiply 8264 by $12\frac{1}{2}$.

WRITTEN PROCESS.

$$\begin{array}{r} 8)826400 \\ \hline 103300 \end{array}$$

3. Multiply 8264 by $112\frac{1}{2}$.

WRITTEN PROCESS.

$$\begin{array}{r} 8)826400 \\ 103300 \\ \hline 929700 \end{array}$$

4. $6476 \times 25 = ?$	5. $6476 \times 125 = ?$
6. $9363 \times 33\frac{1}{3} = ?$	7. $9363 \times 133\frac{1}{3} = ?$
8. $5472 \times 16\frac{2}{3} = ?$	9. $5472 \times 166\frac{2}{3} = ?$
10. $3591 \times 142\frac{6}{7} = ?$	11. $3591 \times 114\frac{2}{7} = ?$
12. $4096 \times 87\frac{1}{2} = ?$	13. $4096 \times 125 = ?$
14. $10824 \times 8\frac{1}{3} = ?$	15. $10824 \times 83\frac{1}{3} = ?$
16. $24320 \times 62\frac{1}{2} = ?$	17. $24320 \times 625 = ?$
18. $4864 \times 31\frac{1}{4} = ?$	19. $4864 \times 312\frac{1}{2} = ?$
20. $24816 \times 125 = ?$	21. $24816 \times 1250 = ?$

CASE II.

When all the figures of the multiplier are alike.

Ex. 1. Multiply 8983725 by 333333.

Ans. 2994572005425.

WRITTEN PROCESS.

$$\begin{array}{r} 8983725000000 \\ 8983725 \\ \hline 3)8983716016275 \\ \hline 2994572005425 \end{array}$$

EXPLANATION.

By Art. 50 the product of 8983725 by 999999 = 8983725000000 — 8983-725, = 8983716016275. Therefore the product by 333333, which is one-third of 999999, must be one-third of 8983716016275, that is 299457200-5425.

Ex. 2. Multiply 98161493 by 7777777.

Ans. 763478202541061.

ANALYSIS.—By Art. 50 the product by 9999999 = 981614930000000 — 98161493, = 981614831838507. Therefore the product by 7777777, which is seven-ninths of 9999999, must be seven times one-ninth of 981614831838507. One-ninth of this is 109068314648723, and 7 times this is 763478202541061.

Rule. *Multiply as if the figures were 9's, and find that part of the product which the figure is of 9.*

Examples for Practice.

3. Multiply 4 3 5 8 6 7 by 4 4 4 4 4.
4. Multiply 3 6 0 7 8 2 by 5 5 5 5.
5. Multiply 6 8 7 6 1 4 7 3 4 5 by 6 6 6 6 6 6 6 6.
6. Multiply 2 1 0 6 8 7 4 6 5 0 7 by 8 8 8 8 8 8 8.
7. Multiply 9 7 5 3 1 8 6 4 2 0 by 3 3 3 3 3.
8. Multiply 6 8 5 3 5 4 6 8 2 by 2 2 2 2 2 2.
9. Multiply 5 1 3 4 8 6 7 by 7 7 7 7 7 7 7 7.

Case III.

When all the figures of the multiplier, except one, are alike.

Ex. 1. Multiply 1324756 by 337333.

Written Process.

```
   1324756000000
         1324756
  ---------------
3)1324754675244
  ---------------
    441584891748
      5299024000
  ---------------
    446883915748
```

Explanation.

Annexing 6 0's to the multiplicand finds the product by a million, $=1324756000000$. Subtracting the multiplicand from this leaves the product by 999999. One third of this product is the product by 333333. Adding to this 4000 times the multiplicand finds the product by 337333.

Rule. *Multiply as if all the figures of the multiplier were alike; and also by the difference between the local values of the single figure and one of the others in the same place. Find the sum of these products when the single figure is more valuable than the other, but the difference of the products when the single figure is less valuable than the other.*

Examples for Practice.

2. $123456 \times 44644 = ?$
3. $527683 \times 555755 = ?$
4. $43864 \times 88898 = ?$
5. $613257 \times 33833 = ?$
6. $520435 \times 669666 = ?$
7. $402763 \times 27222 = ?$

Art. 68. To contract division by multiplication.

CASE I.

When the divisor is a convenient part of a power of 10, or of a convenient number of times that power.

Ex. 1. Divide 234525 by 25. Ans. 9381.

ONE PROCESS.	ANOTHER.	EXPLANATION.
2 3 4 5 2 5 4 9 3 8 1|0 0	2 3 4 5|2 5 4 9 3 8 1	Four times 25, or 100, are contained in four times the dividend as many times as 25 is contained in the dividend. Hence process first finds $\frac{1}{100}$ of $4\times$ dividend.

Again, $\frac{1}{25}$ of the dividend equals 4 times $\frac{1}{100}$ of it. Hence process second finds $4\times\frac{1}{100}$ of dividend.

Rules. I. *Multiply both divisor and dividend by that number which will convert the divisor into the specified power of 10, or into a convenient number of times that power, then divide:* or,

II. *Divide the dividend by the specified power of 10, and multiply the quotient by the number of times that the divisor is contained in that power.*

NOTE.—Some solutions under this case are shorter than the common method, and some are not. These facts are seen by solving by both methods the following

EXAMPLES FOR PRACTICE.

2. $643\div 5=(2\times643)\div(2\times 5),=1286\div 10,= 128\frac{6}{10}$.
3. $643\div15=(2\times643)\div(2\times15),=1286\div 30,= 42\frac{26}{30}$.
4. $643\div35=(2\times643)\div(2\times35),=1286\div 70,= 18\frac{26}{70}$.
5. $643\div45=(2\times643)\div(2\times45),=1286\div 90,= 14\frac{26}{90}$.
6. $643\div55=(2\times643)\div(2\times55),=1286\div110,= 11\frac{76}{110}$.

NOTE.—In the above examples the *true remainders* are half the numerators.

7. $6428\div125=(8\times6428)\div(8\times125),=51424\div1000,$
$=51\frac{424}{1000}$.
8. $6428\div75=(4\times6428)\div(4\times75),=25712\div300,$
$=85\frac{212}{300}$.

9. $6428 \div 225 = (4 \times 6428) \div (4 \times 225), = 25712 \div 900.$
10. $6428 \div 275 = (4 \times 6428) \div (4 \times 275), = 25712 \div 1100.$
11. $35792 \div 250 = (4 \times 35792) \div 1000.$
12. $54768 \div 33\frac{1}{3} = (3 \times 54768) \div 100.$
13. $54768 \div 333\frac{1}{3} =$ what?
14. $73846 \div 166\frac{2}{3} =$ what?
15. $73846 \div 750 =$ what?
16. $85763 \div 142\frac{6}{7} =$ what?
17. $57984 \div 212\frac{1}{2} = (8 \times 57984) \div (8 \times 212\frac{1}{2}), = 463872 \div 1700.$
18. $63142 \div 3250 = (4 \times 63142) \div (4 \times 3250), = 252568 \div 13000$
19. $97531 \div 1750 = (4 \times 97531) \div 7000.$
20. $89763 \div 1125 = (8 \times 89763) \div 9000.$

Case II.

When the divisor is but little less or greater than a power of 10. (See Art. 50.)

Ex. 1. Divide 357684 by 995. Ans. $359\frac{479}{995}$.

Written Process.

$$
\begin{array}{r|r}
995 = 1000 - 5\,)\,357 & 684 \\
1 & 785 \\
 & 5 \\
\hline
1 & 474 \\
 & 5 \\
\hline
359 & 479 \\
\cline{2-2}
 & 995
\end{array}
$$

Explanation.

Dividing by 1000, the quotient is 357, and the remainder 684. But, because the divisor is too large by 5, the remainder is too small by 357 times $5 = 1785$. Dividing this by 1000, the quotient is 1, and the remainder 785. Again, because this divisor is too large by 5, this remainder is too small by $1 \times 5 = 5$. Therefore add 5 to the remainders. The sum of these remainders is 1474. Dividing this by 1000, the quotient is 1, and the remainder 474. Since this divisor was too large by 5, this remainder is too small by $1 \times 5 = 5$. Add 5 to the remainder. The sum of the remainders is 479, and of the quotients 359, making the true quotient $359\frac{479}{995}$.

Ex. 2. Divide 469423 by 1005. Ans. $467\frac{88}{1005}$.

WRITTEN PROCESS.

$$\begin{array}{r|l}
1005 = 1000 + 5\,)\,469 & 423 \\
2 & 345 \\
\hline
467 & 078 \\
 & 10 \\
\hline
467 & \frac{88}{1005}
\end{array}$$

EXPLANATION.

Dividing by 1000, the quotient is 469, and the remainder 423. But, because the divisor is too small by 5, the result is too large by 469 times 5 = 2345. Dividing this by 1000, the quotient is 2, and the remainder 345. Subtracting the 2 from the first quotient, and 345 from the first remainder, the quotient becomes 467, and the remainder 78. But, because the divisor of 2345 was too small by 5, the result subtracted was too large by $2 \times 5 = 10$. Therefore, add 10 to the remainder, making the true remainder 88, and the true quotient $467\frac{88}{1005}$.

Rules I. When the divisor is *less* than a power of 10 by a convenient number called the **complement,** *cut off from the right of the dividend as many figures as the given power has ciphers, multiply the part on the left by the complement, write the result under the dividend, and cut off as before. Do thus till the number of figures in a product is not greater than the number of ciphers in the power. Add the results. If there is any figure carried to the left of the part cut off, multiply it by the complement, and add the result to the part cut off. The sum of the parts on the left is the quotient, and that of the parts cut off is the remainder.*

II. When the divisor is *greater* than a power of 10 by a convenient number, called the **excess,** *cut off from the right of the dividend as many figures as the given power has ciphers, multiply the part on the left by the excess, write the result under the dividend, and cut off as before. Subtract the result from the dividend. Multiply by the excess the left-hand part of the subtrahend, write the product under the remainder, and cut off as before. Add the product to the remainder. Multiply by the excess the left-hand part of the product, write the product under the sum, and cut off as before. Subtract this product from the sum. Do thus till the number of figures in a product is not greater than the number of ciphers in the power. If there is any figure carried to the left of the part cut off in finding the last*

result, multiply it by the excess, and add the product to, or subtract it from the result, according as its turn requires. The part on the left of the final result is the quotient, that on the right is the remainder.

EXAMPLES FOR PRACTICE.

3. Divide 8352250 by 93. Ans. $89809\frac{13}{93}$.
4. Divide 8352250 by 107.
5. Divide 6879057 by 94.
6. Divide 4903501 by 106.
7. Divide 37021604 by 991.
8. Divide 37021604 by 1009. 4693 by 109.
9. Divide 56103729 by 996.
10. Divide 56103729 by 1004.
11. Divide 29180746 by 997.
12. Divide 29180746 by 1003.
13. Divide 87560373 by 9998. By 10002.
14. Divide 87560373 by 1012.
15. Divide 90800671 by 988.
16. Divide 23547698 by 989. By 1011.
17. Divide 3050407 by 999.

CASE III.

When all the figures of the divisor are alike.

Ex. 1. Divide 830547126 by 3333. Ans. $249189\frac{189}{3333}$.

WRITTEN PROCESS.

3333)	830547	126
3		3
9999)	249164	1378
	24	9164
		24
	249189	0566
		1
		3)567
True remainder =		189

EXPLANATION.

Since 3333 is $\frac{1}{3}$ of 9999, we convert the divisor into 9's by multiplying by 3. We also multiply the dividend by 3. Three times the divisor is contained in three times the dividend as many times as the divisor is contained in the dividend. We then divide according to the method in Case II.

The remainder is $\frac{567}{9999}$, which is reduced to 3333d's by dividing by 3, making the true remainder 189, which is $\frac{189}{3333}$.

NOTE.—To multiply by 4444, we would divide both divisor and dividend by 4, and multiply the results by 9; then divide, &c.

Rule. *Divide both divisor and dividend by the common figure of the divisor, and multiply the results by 9, then proceed as in Case II.*

NOTE 1.—When the common figure is 1, merely multiply divisor and dividend by 9; when it is 3, multiply both by 3; when it is 6, divide by 2 and multiply by 3.

NOTE 2.—When there is a remainder, multiply it by the common figure of the divisor, divide the product by 9. The result is the true remainder, which should be placed over the true divisor at the right of the quotient.

EXAMPLES FOR PRACTICE.

2. Divide 876902473 by 2222222.
3. Divide 173241672 by 3333333.
4. Divide 826725840 by 44444.
5. Divide 13060723 by 55555.
6. Divide 32514682 by 666666.
7. Divide 43715685 by 77777.
8. Divide 5321024 by 88888.
9. Divide 7532647 by 111111.

Art. 69. To find the value of a number of the equal parts of a quantity.

Rule. *Multiply the value of one of the parts by the given number of parts.*

EXAMPLES FOR PRACTICE.

1. Find two-thirds of 48. Ans. 32.

ANALYSIS.—One-third of 48 is the quotient of 48 divided by 3, which is 16; and two-thirds of 48 are 2 times 16, which is 32.

2. Find $\frac{2}{3}$ of 132; $\frac{3}{4}$ of 384; $\frac{2}{5}$ of 675; $\frac{3}{5}$ of 3680; $\frac{4}{5}$ of 25425.
3. Find $\frac{5}{6}$ of 247218; $\frac{5}{8}$ of 964832; $\frac{4}{21}$ of 727566.
4. Find 7 thirty-seconds of 315904. Ans. 69104.
5. Find 23 ninety-thirds of 7496265. Ans. 1853915.
6. Find 75 one-hundred-and-fourths of 73008.
7. Find 86 two-hundred-and-sixty-firsts of 97875.
8. Find 125 six-hundred-and-eighty-seconds of 3060134.
9. Find 7 one-thousand-and-sixty-thirds of 6372685.
10. Find $\frac{117}{1111}$ of 143319.

CANCELLATION.

Art. 70. To lesson the labor of dividing, when the divisor and dividend have the same factor.

PRINCIPLE.—*Any part of the divisor is contained in a like part of the dividend as many times as the whole divisor is contained in the whole dividend.*

ILLUSTRATION.

24	is contained in 72	3 times.
$\frac{1}{2}$ of 24, or 12,	is contained in $\frac{1}{2}$ of 72, or 36,	3 times.
$\frac{1}{3}$ of 24, or 8,	is contained in $\frac{1}{3}$ of 72, or 24,	3 times.
$\frac{1}{4}$ of 24, or 6,	is contained in $\frac{1}{4}$ of 72, or 18,	3 times.
$\frac{1}{6}$ of 24, or 4,	is contained in $\frac{1}{6}$ of 72, or 12,	3 times.
$\frac{1}{8}$ of 24, or 3,	is contained in $\frac{1}{8}$ of 72, or 9,	3 times.
$\frac{1}{12}$ of 24, or 2,	is contained in $\frac{1}{12}$ of 72, or 6,	3 times.

To **cancel** a factor from a number is to reject it from that number, leaving its other factors.

Cancellation is a process designed to lesson the labor of dividing by canceling the same factors from the dividend and divisor.

The process of cancellation is usually represented by drawing a line across the number from which a factor is rejected, and writing the other component factor near the number thus crossed. When the other factor is 1, it is not written. Thus, the facts stated in the illustration given above may be represented as follows:—

$$\frac{\cancel{72}^{36}}{\cancel{24}^{12}} = 3 : \quad \frac{\cancel{72}^{24}}{\cancel{24}^{8}} = 3 : \quad \frac{\cancel{72}^{18}}{\cancel{24}^{6}} = 3 :$$

$$\frac{\cancel{72}^{12}}{\cancel{24}^{4}} = 3 : \quad \frac{\cancel{72}^{9}}{\cancel{24}^{3}} = 3 : \quad \frac{\cancel{72}^{6}}{\cancel{24}^{2}} = 3.$$

Ex. 1. A farmer sold 20 pounds of butter at 24 cents a pound, and took his pay in muslin at 15 cents a yard. How ny yards did he receive? Ans. 32.

PROCESS BY CANCELLATION.

$$\frac{{}^{4}\cancel{2}\cancel{0} \times \cancel{2}\cancel{4}^{8}}{\cancel{1}\cancel{5}\atop\cancel{3}} = \frac{4 \times 8}{1} = 32.$$

EXPLANATION.

He received as many yards of muslin as the price of 1 yard, 15 cents, is contained times in the cost of the butter, 20 times 24 cents. Since the factors of 15 are also in 20 times 24, it is not necessary to multiply 24 by 20, and divide the product by 15. Perceiving that 5 is a factor both of 20 and 15, we cross 20 and write its other component factor, 4, near it; we also cross 15, and write its other component factor, 3, near it. Now, perceiving that this factor 3 is also in 24, we cross 24, and write its other component factor, 8, near it; we also cross 3, but need not write its other component factor, 1, near it, because it is always understood in such cases. The result is 4 times 8, or 32, divided by 1. Hence he received 32 yards.

Ex. 2. What is the quotient of 21 times 32, divided by 15 times 12?

PROCESS BY CANCELLATION.

$$\frac{\cancel{2}\cancel{1}^{7} \times \cancel{3}\cancel{2}^{8}}{\cancel{1}\cancel{5}^{5} \times \cancel{1}\cancel{2}^{3}} = \frac{7 \times 8}{5 \times 3} = \frac{56}{15} = 3\tfrac{11}{15}.$$

EXPLANATION.

Perceiving that 3 is a factor common to 21 and 15, we reject it, crossing 21 and 15, and writing the other component 7, near 21, and 5 near 15. Perceiving that 4 is common to 32 and 12, we reject it, crossing 32 and 12, and writing the other component 8 near 32, and 3 near 12. Since no other factor is common to divisor and dividend, the quotient is the product of the remaining factors, 7 and 8, of the dividend, divided by the product of the remaining factors, 5 and 3, of the divisor; that is, 56 divided by 15, equal to $3\frac{11}{15}$.

Rule. *Reject from the divisor and dividend all factors found in both; draw a line across the divisor and dividend, and set their remaining factors near them. Divide the product of the remaining factors of the dividend by the product of the remaining factors of the divisor.*

NOTE 1.—Some prefer to arrange the factors of the dividend on the right of a vertical line, and those of the divisor on the left of it. (See Ex. 3.)

NOTE 2.—If a remaining factor is 1, it *may* be written near the crossed number, but it is neither customary nor necessary, because it will not affect the result when the product of the remaining factors is found.

NOTE 3.—The computer is earnestly recommended to practice cancellation whenever it lessons labor, without obscuring the written process.

Examples for Practice.

3. Divide 27×32 by 12×21.

First Method.

$$\frac{\overset{9^3}{\cancel{27}} \times \overset{8}{\cancel{32}}}{\underset{\cancel{3}}{\cancel{12}} \times \underset{7}{\cancel{21}}} = \frac{3 \times 8}{7} = 3\tfrac{3}{7}$$

Second Method.

$$\begin{array}{cc|cc} \cancel{3} & \cancel{12} & \cancel{27} \; \cancel{9} & 3 \\ 7 & \cancel{21} & \cancel{32} & 8 \end{array}$$

$$7\overline{)3 \times 8 = 24}$$
$$3\tfrac{3}{7}$$

4. Divide 35×40 by 28×25.
5. Divide $33 \times 45 \times 42$ by 28×55. Ans. $40\frac{1}{2}$.
6. Divide $24 \times 25 \times 30 \times 40$ by $45 \times 36 \times 20$. Ans. $22\frac{2}{9}$.
7. Divide $18 \times 20 \times 22 \times 24$ by $33 \times 12 \times 15$.
8. Divide $8 \times 14 \times 32 \times 12 \times 6$ by $24 \times 21 \times 16$.
9. Divide $15 \times 18 \times 50 \times 60 \times 49$ by $14 \times 21 \times 120 \times 45$.
10. Divide $144 \times 324 \times 500$ by 1728×125. Ans. 108.
11. Divide $51 \times 8 \times 20 \times 39$ by $26 \times 34 \times 45$.
12. Divide $36 \times 24 \times 150 \times 121$ by $33 \times 120 \times 55 \times 18$.

Art. 71. Numbers expressed in other scales of notation than the decimal are divided on a similar plan.

Ex. 1. Divide *nine hundred forty-three* by *forty-one* in senary notation. Ans. 35, senary; = 23 decimal.

Written Process.

Divisor. (36's)	(6's)	(1's)		Dividend. (216's)	(36's)	(6's)	(1's)		Quotient. (6's)	(1's)
1	0	5	)	4	2	1	1	(	3	5
				3	2	3				
					5	4	1			
					5	4	1			

Explanation.

If we *think* in decimal notation, and reduce results to senary, we compute thus:—One 36 and 5 are contained in four 216's and two 36's three 6's times. Write 3 in the quotient, and multiply the divisor by it, (See Art. 54,) making the first subtrahend 3 (216's) 2 (36's) 1 (6.) Subtract, (See Art. 39,) and the remainder is 5 (36's) 4 (6's.) Bring down the 1, making the new dividend 5 (36's) 4 (6's) 1. Dividing this, 1 (36) is contained in 5 (36's) 5 times. Write 5 in the quotient, and multiply the divisor by it, making the subtrahend 541. Subtracting 0 remains, and the complete quotient is 3 (6's) 5, equal to 23 in decimal notation.

CHAPTER VII.

PRINCIPLES AND PROBLEMS IN THE PRECEDING RULES.

NOTATIONS.*

Art. 72. The decimal notation has been so thoroughly incorporated into the languages and customs of civilized nations, that they have no terms to aid in pronouncing numbers written in a notation of another radix, nor can they think well in any other. (See Arts. 26, 27, 28.) Since no advantage would result from completing and adopting other notations, their discussion has no practical value, except the development of the fundamental philosophy of all systems, the decimal included, based upon a definite scale.

Reduction of Notations.

Art. 73. Reduction of notations is the process of changing a number expressed in one notation to an equivalent number in another.

Art. 74. To reduce a decimal number to another scale.

Ex. 1. Reduce *four thousand five hundred and twenty-three* to the ternary scale. Ans. 20012112.

---

* To be omitted, if desired. (See Art. 34, note.)

Written Process.

3) 4 5 2 3 *units.*

3) 1 5 0 7 *threes,* + 2 *units.*

3) 5 0 2 *nines,* + 1 *three.*

3) 1 6 7 (2 7's), + 1 *nine.*

3) 5 5 (8 1's), + 2 (2 7's).

3) 1 8 (2 4 3's), + 1 (8 1).

3) 6 (7 2 9's), + 0 (2 4 3).

2 (2 1 8 7's), + 0 (7 2 9).

Ternary Number Constructed.

(2 1 8 7's)	(7 2 9's)	(2 4 3's)	(8 1's)	(2 7's)	(9's)	(3's)	(*units*).
2	0	0	1	2	1	1	2

Proof.

2×2187	=	4374.
1×81	=	81.
2×27	=	54.
1×9	=	9.
1×3	=	3.
2×1	=	2.
Amount	=	4523.

Rule. *Divide the number by the radix of the required scale. Divide the quotient by the radix, and so proceed till the quotient is less than the radix. To the last quotient annex in order the remainders from the last to the first, writing a cipher when there is no remainder. The number thus constructed is in the scale required.*

Examples for Practice.

Reduce

2. Two thousand twenty-nine to the tenary scale. Ans. 2210011.
3. 9725 to the quaternary scale. Ans. 2113331.
4. 8759 to the quinary scale.
5. 4628 to the senary scale.
6. 11853 to the septenary scale. Ans. 46362.
7. 123457 to the nonary scale.
8. 13579 to the octary scale.
9. 23456 to the undenary scale.

53761 to the duodenary scale. Ans. 27141.

Art. 75. To reduce to the decimal scale a number in another scale.

Ex. 1. Reduce 2 (25's) 3 (5's) 4 (1's) from the quinary to the decimal scale. Ans. 69.

FIRST METHOD.

$$\begin{array}{l} 2\ 3\ 4 \\ \underline{5} \\ 1\ 3 \\ \underline{5} \\ 6\ 9 \end{array}$$

ANALYSIS.—Since one 25 equals 5 (5's), two (25's) equal 2 times 5 (5's), that is, 10 (5's), and the 3 (5's) given make 13 (5's). Since one 5 is 5 units, 13 (5's) are 13 times 5 units, that is, 65 units, and the 4 units given make 69 units.

SECOND METHOD.

$$\begin{array}{rcrl} 2 \times & 25 = & 50 \\ 3 \times & 5 = & 15 \\ 4 \times & 1 = & \underline{4} \\ & & 69 \end{array}$$

Rule. *Multiply the left-hand figure of the number by its radix, and to the product add the next figure. Multiply this sum by the radix, and to the product add the next figure. So proceed to the last figure. The last sum is the number in the decimal scale.*

EXAMPLES FOR PRACTICE.

Reduce to the decimal scale.

2. 4231 in the quinary scale. Ans. 566.
3. 12021 in the ternary scale.
4. 2103213 in the quaternary scale.
5. 32412 in the senary scale.
6. 46362 in the septenary scale.
7. 76543 in the octary scale.
8. 23456 in the nonary scale.
9. 1011001 in the binary scale.
10. 23154 in the undenary scale.
11. 30142 in the duodenary scale.

Art. 76. To reduce a number from one non-decimal scale to another.

Rule. *Reduce the number to the decimal scale, then that result to the required scale.*

EXAMPLES FOR PRACTICE.

Reduce

1. 3102 in the quaternary scale to the ternary scale. Ans. 21210.
2. 4052 in the senary scale to the quinary scale. Ans. 12041.
3. 1234 in the octary scale to the binary scale.
4. 3456 in the septenary scale to the nonary scale. Ans. 1656.
5. 7605 in the octary scale to the nonary scale.
6. 101001 in the binary scale to the senary scale.

Art. 77. To add, subtract, multiply, or divide in a non-decimal scale.

Rule. *Proceed as in the decimal scale, using the given radix as 10 is used in similar cases in the decimal scale.*

NOTE.—Examples of these operations have already been presented in Articles 34, 39, 54, and 71.

EXAMPLES FOR PRACTICE.

1. Add 53, 47, 253, 741, and 124 in octary notation.
2. From 131, take 55 expressed in senary notation.
3. Multiply 53 by 36 in quaternary notation.
4. Divide 23410 by 32 in quinary notation.

VARIATION.

Art. 78. One quantity varies *directly* as another, when it increases proportionally as the other increases, and decreases proportionally as the other decreases. Thus, if the price of a unit of any article is fixed, the cost of any number of articles varies directly as their number; and in general, the quantity of effect varies directly as the quantity of cause.

ILLUSTRATION.—At 10 cents per pound, 2 pounds of sugar cost 2 times 10 cents, 3 pounds cost 3 times 10 cents, &c. Again, at 25 miles per hour, a train moves, in 2 hours, 2 times 25 miles, in 3 hours 3 times 25 miles, &c.; that is the distance traveled varies directly as the time.

One quantity varies *inversely* as another when it increases proportionally as the other decreases, and decreases proportionally as the other increases. Thus, the time occupied in traveling a certain distance varies inversely as the velocity; that is, with twice a given velocity, the distance would be traveled in half the time; with three times the given velocity, in one-third of the time.

ILLUSTRATION.—At 2 miles per hour, it would require 6 hours to walk 12 miles; but, at twice 2 miles per hour, or 4 miles per hour, it would require only one-half of 6 hours, that is, 3 hours.

Art. 79. In respect to their variation, the quantities of a problem are either *variables* or *constants*.

A **variable** is a quantity which may have different values without altering the form of the problem.

A **constant** is a quantity which does not vary with the other quantities with which it is connected.

Art. 80. The sign of variation is the symbol $\propto$, placed between the symbols which represent the varying quantities. Thus, if C is a symbol for cost, and N for the number of articles, the fact that cost varies directly as quantity is expressed by $C \propto N$. If D represents distance traveled, and T the time of travel, the fact that distance varies directly as time is expressed by $D \propto T$.

Inverse variation is represented by writing the second member of the expression as a fraction having the numerator 1. Thus, if V represents velocity and T time, the fact that the time of traveling a certain distance varies inversely as the velocity is expressed by

$$T \propto \frac{1}{V}$$

This, interpreted, means that, if the value of T is taken as twice what it was in a certain case, that of V must be half what it was in that case; if T is taken as 3 times what it was, V must be one-third of what it was, &c.

RELATIONS BETWEEN FACTORS AND PRODUCT.

Art. 81. *The product varies directly as the factors.*

ILLUSTRATION.—Since any factor may be considered the multiplier, doubling a factor doubles the number of times the other factor is taken, halving a factor halves the number of times the other factor is taken, &c.

Art. 82. *The product is constant when one factor varies inversely as the other.*

ILLUSTRATION.—The product of a factor, taken a certain number of times, is the same as that of twice that factor, taken half as many times, or as one-third of that factor, taken three times as many times, &c.

RELATIONS BETWEEN DIVIDEND, DIVISOR, AND QUOTIENT.

Art. 83. *When the divisor is constant, the quotient varies directly as the dividend.*

ILLUSTRATION.—The same divisor is contained in twice the dividend twice as many times; in three times the dividend three times as many times; in half the dividend half as many times, &c.

Art. 84. *When the dividend is constant, the quotient varies inversely as the divisor.*

ILLUSTRATION.—In the same dividend twice the divisor is contained half as many times; three times the divisor, one-third as many times, one-half of the divisor, twice as many times, &c.

Art. 85. *When the dividend and divisor vary proportionally, the quotient is constant.*

ILLUSTRATION.—Twice the divisor is contained in twice the dividend as many times as the divisor is contained in the dividend. The same is true of three times the divisor in three times the dividend, one-half of the divisor in one-half of the dividend, &c.

PROBLEMS IN THE FUNDAMENTAL RULES.

Art. 86. Addition, Subtraction, Multiplication, and Division are called the fundamental rules of Arithmetic, because every arithmetical process depends upon one or more of them. These four operations, are strictly speaking, reducible to two; namely, *Addition*, which has for its object the

increase of a quantity, and *Subtraction*, which has for its object the *decrease* of a quantity. Multiplication may be considered as only a convenient method of obtaining the results of certain additions, and Division as only a convenient method of obtaining the results of certain subtractions.

Problems in Addition.

Art. 87. The fundamental problem of Addition is

Given the parts of an unknown quantity, required to find that quantity.

The specific forms of this problem are

1. Given two or more numbers to find their sum.
2. Given the parts to find the whole.
3. Given the less of two numbers and their difference, to find the greater.

Illustrations.

1. Find the sum of 483679, 8007346, 800, 800400, and 7960432.
2. Find the sum of 809, 4768407, 80004324, 7960434, and 7764383.
3. In a farm, A owned 640 acres, B 720, and C 872. How many acres in the whole farm?
4. The smaller of two numbers is 7963, their difference is 728. What is the greater number?

Problems in Subtraction.

Art. 88. The fundamental problem of Subtraction is

Given the whole of a certain quantity, and one of its two component parts, required to find the other component part.

The specific forms of this problem are

1. Given two numbers, to find their difference.
2. Given the sum of two numbers, and one of them, to find the other.
3. Given the whole of a quantity, and all the parts except one, to find that part.

4. Given the greater of two numbers, and their difference, to find the less.

ILLUSTRATIONS.

1. From 78400038 take 46784.
2. The sum of two numbers is 8496384, one of the numbers is 79849; what is the other?
3. A, B, C, and D own a farm containing 7644 acres. A owns 763 acres, B 972, D 2464, and C the remainder. How many acres does C own?
4. The greater of two numbers is 48796, and their difference is 40006. What is the other number?

PROBLEMS IN MULTIPLICATION.

Art. 89. The fundamental problem of Multiplication is *Given a quantity, and the number of quantities of the same kind and value required to find the sum of that number of those quantities.*

The specific forms of this problem are

1. Given equal quantities, and their number, to find their sum.

2. Given one of the equal parts of a quantity, and their number, to find that quantity.

3. Given the number of causes, and the value of each, to find the effect.

ILLUSTRATIONS.

1. B has 486 sheep in each of 13 fields. How many sheep has he?
2. An estate was divided among 43 heirs, each receiving $2748. What was the estate worth?
3. If 16 men build a wall in 35 days, how long will it take 1 man to build it?

PROBLEMS IN DIVISION.

Art. 90. The fundamental problems of Division are

1. *Given a quantity, and one of its equal parts, required to find the number of such parts.*

2. *Given a quantity, and the number of its equal parts, required to find one of those parts.*

The specific forms of these problems are

1. Given the sum of equal quantities, and one of them, to find their number.

2. Given the sum of equal quantities, and their number, to find one of them.

3. Given the effect of two causes, and one of them, to find the other.

4. Given the sum of two or more quantities, and their number, to find their *average*, or *mean*.

ILLUSTRATIONS.

1. A man gave to each of his children \$720, and to all of them he gave 6480 dollars; how many children had he?

2. A man put 1260 bushels of apples into 20 bags; how many bushels did he put in each?

3. If 1 man can build 725 rods of fence in 29 days, how long will it take 25 men to build the same amount?

4. If a ship sails 1189 miles in 29 days, what is the average rate per day?

Art. 91. The *average*, or *mean*, of two or more numbers is that number which, expressed as many times as there are numbers, would amount to their sum. It implies that the sum is considered as distributed into as many equal quantities as there are numbers. It is a popular method of dealing with unequal quantities, when they are considered together.

Art. 92. To find the average, or mean, of numbers.

Rules.—I. *Divide their sum by their number.* Or

II. *Divide the sum of the excesses of the other numbers over the least by the number of numbers, and add the quotient to the least number.* Or

III. *Divide the sum of the deficiencies, by which the other numbers are less than the greatest, by the number of numbers, and subtract the quotient from the greatest number.*

EXAMPLES FOR PRACTICE.

1. Bought one cow for 36 dollars, and 2 others for 30 dollars a piece. What was their average cost? Ans. 32 dollars.

FIRST PROCESS.		SECOND PROCESS.		THIRD PROCESS.	
1, at 36, cost	36	1, excess	6	1, deficit	0
2, " 30, "	60	2, "	0	2, "	12
3	)96	3	)6	3	)12
av. cost =	32	av. excess =	2	av. def. =	4
		30 + 2 =	32	36 − 4 =	32

2. Find the average of 4, 8, 12, and 16. Ans. 10.
3. Find the average of 9, 12, 15, and 18. Ans. $13\frac{2}{4}$.
4. Find the average of 26, 37, 48, and 59. Ans. $42\frac{2}{4}$.
5. A's age is 45, B's 30, C's 35, D's 60, E's 70 years. What is the average of their ages? Ans. 48 years.
6. The weights of some hogs are as follows: 250, 320, 275, 322, 415, 213, 244, 209, and 195 pounds. What is the average of their weights?
7. A man sold goods in six days to the following amounts: $80, $75, $92, $63, $210, $193. What did his sales average per day?
8. Seven houses are worth, respectively, $10000, $12000, $8500, $7525, $4260, $4180, and $3200. What is their average value?

Art. 93. Given the sum and difference of two numbers, to find the numbers.

Ex. 1. The sum of two numbers is 40, and their difference is 14. What are the numbers? Ans. 13 and 27.

ANALYSIS.—First, since 40 = the greater + the less, 40 + 14 = the greater + the less + 14. But the less + 14 = the greater. Therefore 40 + 14 = twice the greater. Again, since 40 = the less + the greater, 40 — 14 = the less + the greater — 14. But the greater — 14 = the less. Therefore 40 — 14 = twice the less.

Rules.—I. *To the sum add the difference, and divide the result by 2; the quotient will be the greater number.* Or

II. *From the sum take the difference, and divide the result by 2; the quotient will be the less number.*

Examples for Practice.

2. The sum of two numbers is 56, and their difference 16. What are the numbers?

3. The sum of two numbers is 81, and their difference 23. What are the numbers?

4. The sum of two numbers is 843, and their difference is 165. What are the numbers?

Art. 94. Given the sum of more that two numbers, and their differences, to find the numbers.

Rule.—*From the sum take the difference between the least and every other number, and divide the remainder by the number of numbers; the quotient will be the least number. To the least number add the difference between it and another; the sum is that number.*

Note.—The pupil should show the reason for the rule, in analyzing the solutions of examples.

Examples for Practice.

1. The sum of 3 numbers is 557; the difference between the first, or smallest, and the second is 48, and the difference between the first and third is 113. What are the numbers?

2. The sum of three numbers is 846; the difference between the first, or smallest, and the second is 15, and the difference between the second and third is 57. What are the numbers?

SYNOPSES OF FUNDAMENTAL PROCESSES.

1. Synopsis of Notations.

Scales.	Operations.
Binary. Ternary. Quaternary. Quinary. Senary. Septenary. Octary. Nonary. Denary, or Decimal. Undenary, or Undecimal. Duodenary, or Duodecimal, &c.	Reduction. Addition. Subtraction. Multiplication. Division.

2. Synopsis of the Fundamental Rules.

Rules.	Application to	Points Discussed.	Kinds.	Branches.
Addition. Subtraction. Multiplication. Division.	Simple Numbers.	Definitions.		
		Purposes.		
		Signs.		
		Terms.		
		Principles.		
		Operations.	Common. Special.	Rule. Proof. Exercises.
		Problems.	Fundamental. Specific.	
		Variations.	Direct. Inverse.	

CHAPTER VIII.

PROPERTIES AND RELATIONS OF NUMBERS.

Art. 95. The **properties** of numbers are the qualities which are inseparable from them.

Art. 96. The **integral factors** of a number are such whole numbers as multiplied together, produce that number. They are often called simply *factors* of the number. In this sense

A **factor** of a whole number is any whole number which when multiplied by another, will produce the given number.

An **exact divisor** of a number is a divisor which is contained in that number a whole number of times. Thus, 7 is an exact divisor of 7, 14, 21, &c. Such a divisor is often called simply **a divisor**, or **measure** of the number divided.

One number is said to be **divisible** by another, when it contains that other a whole number of times. Thus, 35, 42, 49, &c., are said to be divisible by 7. Such a number is said to be *divided*, or *measured* by its divisor.

Division is said to be **exact**, or **without a remainder**, when the quotient is a whole number.

NOTE.—Strictly speaking, all complete division is exact, and every number is divisible by another equal to, or less than, itself. But convenience restricts the use of the terms *exact* and *divisible* to cases in which the quotient is an integer.

Art. 97. A **multiple** of a given whole number is any whole number of which it is a factor. It is called a *multiple* because it is considered as produced by multiplying the given number. Thus, 35 is a multiple of 7, because 35 may be produced by multiplying 7 by 5. In like manner 42 is a multiple of 7 by 6.

Any number has an unlimited number of multiples, because it may be multiplied by any number of integral multipliers. Thus, the multiples of 7 are 14, 21, 28, 35, 42, &c., without limit.

Art. 98. In reference to their being produced by integral factors, whole numbers are either *prime* or *composite.*

A **prime number** is a number which cannot be produced by other integral factors than itself and unity. Thus, 3, 11, 17, 23, 29, &c., are prime numbers.

A **composite number** is a number produced by other integral factors than itself and unity. (See Art. 45.)

Numbers are **relatively prime** or **prime to each other** when they have no common factor, except unity. Thus, 6 and 25 are prime *to each other,* though they are not prime numbers separately.

NOTE.—By the definitions, all composite numbers are *prime to all prime numbers,* and all primes are primes to each other.

The **prime factors** of a number are those factors of it which are prime numbers. Thus, the prime factors of 100 are $2 \times 2 \times 5 \times 5$, though 100 may be produced by the composite factors 4×25.

NOTE.—According to the definition of a factor, (Art. 96,) it is plain that 1 and any number are factors of that number, because it may be produced by the multiplication of itself by 1. But, unless otherwise specially stated, these are not reckoned in the list of factors. Thus, although $10 \times 1 = 10$, and $1 \times 2 \times 5 = 10$, yet, without a special statement, we consider the factors of 10 as only 2 and 5.

Art. 99. In reference to their divisibility by 2, whole numbers are either *odd* or *even.*

An **odd number** is a number which is not divisible by 2. Thus, 1, 3, 5, 7, 9, 11, 13, &c., are odd numbers.

An **even number** is a number which is divisible by 2. Thus, 2, 4, 6, 8, 10, 12, 14, &c., are even numbers.

NOTE.—Some fanciful and useless classifications of numbers by the ancients were the following:—

A *perfect number* is equal to the sum of all its factors. Thus, 6 is *perfect according to* this definition, being equal to $1 + 2 + 3$. Only the

following have been discovered:—6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128, 2417851639228158837784576, and 9903520134282971830448816128.

An *imperfect number* is not equal to the sum of all its factors.

An *abundant number* is less than the sum of all its factors.

A *defective number* is greater than the sum of all its factors.

DIVISIBILITY OF INTEGERS.

Art. 100. In attempting to find the factors of numbers, a knowledge of the following propositions will save labor:—

Proposition I.—*A factor of a number is a divisor of it.*

DEMONSTRATION.—Since a whole number of times a factor of a number equals that number, that factor is contained in the number a whole number of times, and is, therefore, a divisor of the number.

COROLLARY.—A number is a divisor of all its multiples.

Proposition II.—*A divisor of a number is a factor of it.*

DEMONSTRATION.—Since an exact divisor of a number is contained in it a whole number of times, a whole number of times the divisor equals the number; therefore, a divisor of a number is a factor of it.

NOTE.—Each of these two propositions is the *converse*, or opposite, of the other. The converse of a true proposition is not always true. In the following propositions the converse of a proved proposition is sometimes merely stated without proof, as sufficiently apparent.

Proposition III.—*A factor, or divisor, of a number is a factor, or divisor, of any multiple of that number.*

DEMONSTRATION.—In the equation, $15 = 3 \times 5$, the factors of 15 are 3 and 5. Multiplying both sides by any number, as 2, (Axiom 3,) we have $2 \times 15 = 2 \times 3 \times 5$, in which the factors of 2×15 are $2 \times 3 \times 5$. Therefore, 3 and 5, the factors of 15, are also factors of 2 times 15.

Proposition IV.—*A factor, or divisor, of each of two or more numbers is a factor, or divisor of their sum.*

DEMONSTRATION.—If each number contains the same factor, their sum must contain that factor as many times as all the numbers contain it, which must be a whole number of times, because its parts are whole numbers. Thus, $6 = 2 \times 3$, and $15 = 5 \times 3$: therefore, $(6 + 15) = (2 \times 3) + (5 \times 3), = (7 \times 3)$.

NOTE.—A factor, or divisor, of each of two or more numbers is said to be *common to them*, and is called a *common factor*, or *common divisor* of those numbers.

Proposition V.—*A factor, or divisor, of one of two numbers, and not of the other, is not a factor, or divisor, of this sum.*

DEMONSTRATION.—If one number contains the divisor a whole number of times, and the other does not, their sum contains the divisor as many times as both numbers contain it, which must be a mixed number because it is a sum of the whole and a mixed number. Thus, $15 = 5$ times 3, and $7 = 2\frac{1}{3}$ times 3; hence $(15 + 7) = (5 + 2\frac{1}{3} =)\ 7\frac{1}{3}$ times 3.

Proposition VI.—*A factor, or divisor, of each of two unequal numbers is a factor, or divisor, of their difference.*

DEMONSTRATION.—If each of two unequal numbers contains the divisor a whole number of times, the greater must contain that divisor as many times more than the less contains it as the difference between the two numbers contains it. But the greater contains it a whole number of times more than the less contains it, because both numbers of times are whole numbers. Therefore, the difference between the two numbers contains the divisor a whole number of times. Thus, $21 = 7 \times 3$, and $15 = 5 \times 3$: therefore $21 - 15$, or $6, = (7 \times 3) - (5 \times 3), = 2 \times 3$.

Proposition VII.—*A factor, or divisor, of one of two unequal numbers, and not of the other, is not a factor, or divisor, of their difference.*

DEMONSTRATION.—If the number of times that one number contains the divisor is integral, and the number of times that the other number contains it is fractional, then the difference of these numbers of times is fractional. But this difference is equal to the number of times that the divisor is contained in the difference of the numbers. Therefore, a factor, or divisor, of one of two unequal numbers, and not of the other, is not a factor, or divisor of their difference. Thus, $15 = 5$ times 3, and $7 = 2\frac{1}{3}$ times 3 : hence, $15 - 7$, that is, $8, = (5 \times 3) - (2\frac{1}{3} \times 3), = 2\frac{2}{3} \times 3$.

Proposition VIII.—*A number which is not a factor, or divisor, of either of two numbers, may, or may not, be a factor, or divisor, of their sum, or of their difference.*

NOTE.—Let the student demonstrate and illustrate the statements of *this proposition.*

Proposition IX.—*A number whose right-hand part is one or more ciphers, is divisible by that power of 10 which has the same number of ciphers.*

DEMONSTRATION.—By the nature of the decimal notation, that part of a number whose right-hand significant figure is in a certain place expresses so many units of the local value of that place. Thus, if the right-hand significant figure is in the tens' place, the part of the number that ends there expresses so many tens, that is, so many times 10: if the right-hand significant figure is in the hundreds' place, the part that ends there expresses so many hundreds, or times 100. Therefore, the number whose right-hand part is one cipher is a multiple of 10, and, consequently, divisible by 10: and a number whose right-hand part is two ciphers is a multiple of 100, and divisible by 100, &c. (See Proposition I.)

ILLUSTRATION.—The number 290 equals 29 *tens*, or 29 times 10; 2900 equals 29 *hundreds*, or 29 times 100, &c.

Converse.—*No number is divisible by a power of 10, unless its right-hand part has as many ciphers as that power.*

Proposition X.—*A whole number whose right-hand figure is 0, 2, 4, 6, or 8, is divisible by 2.*

DEMONSTRATION.—If the right-hand figure is 0, the number is divisible by 10, according to Proposition I, and because 10 is 5 times 2, the number must be divisible by 2, according to the First Principle. If the right-hand figure is 2, 4, 6, or 8, the number is composed of *tens*, plus 2, 4, 6, or 8 units. Now 2 is a factor of the tens, and also of the units. Hence, 2 is a factor of the *sum* of the tens and units, that is, of the *whole number*, according to Prop. IV.

Converse.—*No number is divisible by 2, unless its right-hand figure is 0, 2, 4, 6, or 8.*

COROLLARY.—Hence all numbers ending in 0, 2, 4, 6, or 8, are *even numbers;* and all numbers ending in 1, 3, 5, 7, or 9, are *odd numbers.*

Proposition XI.—*A number is divisible by 4 when the number expressed by its two right-hand figures is divisible by 4.*

DEMONSTRATION.—All the part at the left of the tens' place expresses hundreds, that is, either 4 times 25, or a multiple of 4 times 25. Hence that part is divisible by 4, according to the First Principle. If, also, the number expressed by the two remaining figures is divisible by 4, then the sum of these numbers, that is, the whole number, is divisible by 4.

ILLUSTRATION.—$728 \div 4 = (700 + 28) \div 4, = 175 + 7 = 182.$

Converse.—*No number is divisible by 4, unless the number expressed by its two right-hand figures is divisible by 4.*

COROLLARY.—In the same manner it may be shown that a number is divisible by any power of 2, when the number expressed by as many of its right-hand figures as there are units in the index of that power is either 0, or is divisible by that power. Thus, if n represents any number,

A number of *tens*, $+ n \times 2^1$, is divisible by 2^1, or 2.
A number of *hundreds*, $+ n \times 2^2$, is divisible by 2^2, or 4.
A number of *thousands*, $+ n \times 2^3$, is divisible by 2^3, or 8.
A number of *ten-thousands*, $+ n \times 2^4$, is divisible by 2^4, or 16.
A number of *hundred-thousands*, $+ n \times 2^5$, is divisible by 2^5, or 32.
Etc. Etc. Etc.

ILLUSTRATIONS.—$5752 \div 8 = (5000 + 752) \div 8, = 625 + 94, = 719.$
$154080 \div 16 = (150000 + 4080) \div 16, = 9375 + 255 = 9630.$
Etc. Etc. Etc.

Proposition XII.—*A number is divisible by 5 when its right-hand figure is 0 or 5.*

DEMONSTRATION.—If the right-hand figure is 0, the number expresses *tens*, that is, either 2 times 5, or a multiple of 2 times 5. Hence it is divisible by 5. If the right-hand figure is 5, the *units* are 5, and the *tens* are divisible by 5, according to Proposition III, and therefore their sum, that is the whole number, is divisible by 5.

Converse.—*No number is divisible by 5, unless its right-hand figure is either 0 or 5.*

COROLLARY.—In the same manner it may be shown that a number is divisible by any power of 5, when the number expressed by as many of its right-hand figures as there are units in the index of that power is either 0, or is divisible by that power. Thus, if n represents any number,

A number of *tens*, $+ n \times 5^1$, is divisible by 5^1, or 5.
A number of *hundreds*, $+ n \times 5^2$, is divisible by 5^2, or 25.
A number of *thousands*, $+ n \times 5^3$, is divisible by 5^3, or 125.

ILLUSTRATIONS.—$635 \div 5 = (630 + 5) \div 5, = 126 + 1, = 127.$
$675 \div 25 = (600 + 75) \div 25, = 24 + 3, = 27.$
$4375 \div 125 = (4000 + 375) \div 125, = 32 + 3, = 35.$
Etc. Etc. Etc.

Proposition XIII.—*A number is divisible by a smaller composite number when it is divisible by each factor of that number.*

DEMONSTRATION.—If every factor of one number is also a factor of a larger number, then the product of all the factors of the smaller number is a factor of the larger, and therefore the larger number is divisible by the smaller.

ILLUSTRATION.—If 30, which is equal to $2 \times 3 \times 5$, is divisible by 2 and by 3, it is also divisible by 6, which is equal to 2×3.

Converse.—*No number is divisible by a composite number, unless it is divisible by each factor of that number.*

ILLUSTRATION.—The number 14, which is equal to 2×7, is not divisible by 10, which is equal to 2×5, because 2×7 is not divisible by 5.

COROLLARY 1.—*If an even number is divisible by an odd number, it is also divisible by twice that odd number.*

DEMONSTRATION.—An even number is divisible by 2. If it is also divisible by another number, it must be divisible by the product of 2 and that number.

COROLLARY 2.—*An odd number is not divisible by an even number.*

DEMONSTRATION.—Since 2 is a factor of an even number, but not of an odd number, an odd number is not divisible by every factor of an even number, and therefore, is not divisible by an even number.

Proposition XIV.—*A number, expressed in the decimal notation, is divisible by 9, when the sum of its figures is divisible by 9.*

DEMONSTRATION.—Since $10 = 9 + 1$, the local value of the figure in the *tens'* place is 9 times that figure, plus once that figure. Since $100 = 99 + 1$, the local value of the figure in the *hundreds'* place is 99 times that figure, plus once that figure, and so on. Now, since 9, 99, &c., are each divisible by 9, the whole number is divisible by 9 if the remaining part of the local values, that is, once each figure, is divisible by 9.

Converse.—*No number expressed in the decimal notation, is divisible by 9, unless the sum of its figures is divisible by 9.*

COROLLARY 1.—A number expressed in the decimal notation will, when divided by 9, leave the same remainder as the sum of its figures will leave when divided by 9.

NOTE.—Advantage is taken of this fact in applying tests for correctness of computations in addition, subtraction, multiplication, and divi-

sion. (See Proofs by casting out 9's.) These tests, called *proof by casting out nines*, use the following facts:—

1. In addition, the excess over the nines of the numbers added must be equal to the excess over the nines of the sum, which equals them all.

2. In subtraction, the excess over nines of the subtrahend and remainder must be equal to the excess over nines in the minuend, which equals them.

3. In multiplication, the excess over nines in the product is equal to the excess over nines in the product of the excesses of the factors.

4. In division, the excess over nines of the dividend is equal to the excess over nines in the product of the excesses of divisor and quotient plus the excess in the remainder.

In these methods of proof it is assumed that the computation is correct if the excesses of nines are the same as they would be in correct computation. But, since excesses of nines may be the same in some incorrect computations as in correct, this conclusion is not reliable.

COROLLARY 2.—Since 3 is a measure of 9, the statement of corollary 1 is as true of 3 as it is of 9.

COROLLARY 3.—In any notation having a regular scale, this property belongs to the number which is 1 less than the radix. Thus, in the *quinary* notation it belongs to 4, in the *octary* notation it belongs to 7, &c.

COROLLARY 4.—If the sum of the figures of an *even* number is divisible by 6, the number is divisible by 6. Since it is *even*, it is divisible by 2, and, if the sum of its figures is divisible by 6, it is divisible by 3, a factor of 6. Hence it is divisible by 2 times 3, or 6.

Proposition XV.—*A number, expressed in the decimal notation, is divisible by 11, when the sum of the figures in the odd places is equal to the sum of the figures in the even places, or when the difference of these sums is divisible by 11.*

DEMONSTRATION.—Since 1, 100, 10000, &c., that is, unities of the odd places, when divided by 11, all give 1 remainder, they are each 1 more than a multiple of 11. Since 10, 1000, 100000, &c., that is, unities of the even places, when divided by 11, all give 10 remainder, they are each 1 less than a multiple of 11. Hence the figures of any number, being multiples of these unities, signify in the odd places a multiple of 11, plus as many units as the figures express, and, in the even places a multiple of 11, minus as many units as the figures express. Therefore the whole number is a multiple of 11, plus the sum of its figures in the odd places, minus the sum of its figures in the even places. If, therefore, these sums are equal, or if they differ by a multiple of 11, the whole number is a multiple of 11, and, consequently, is divisible by 11.

NUMERICAL ILLUSTRATION.

$1 =$	$0 \times 11 + 1$,	therefore	$9 =$	$0 \times 11, +9$
$10 =$	$1 \times 11 - 1$,	"	$80 =$	$8 \times 11, -8$
$100 =$	$9 \times 11 + 1$,	"	$700 =$	$63 \times 11, +7$
$1000 =$	$91 \times 11 - 1$,	"	$6000 =$	$546 \times 11, -6$
$10000 =$	$909 \times 11 + 1$,	"	$50000 =$	$4545 \times 11, +5$
$100000 =$	$9091 \times 11 - 1$,	"	$400000 =$	$36364 \times 11, -4$
$111111 =$	10101×11		$456789 =$	$41526 \times 11, +3$

In the first of these numbers, namely, 111111, the sum of the figures in the odd places is equal to the sum of the figures in the even places, and the number is a multiple of 11. In the second number, namely, 456789, the sum of the figures in the even places is not equal to the sum of the figures in the odd places, and the difference of these sums, namely 3, is not a multiple of 11, and the whole number is not a multiple of 11.

COROLLARY 1.—A number, expressed in the decimal notation, will, when divided by 11, leave the same remainder as the sum of its figures in the even places taken from the sum of its figures in the odd places, or from 11 more.

ILLUSTRATION.—In 2637485, the sum of the figures in the odd places is 14; of those in the even places it is 21. Increase 14 by 11, making 25 : 25 — 21= 4, which is also the remainder obtained by dividing 2637485 by 11.

NOTE.– Advantage is taken of this fact to test the correctness of computations in the fundamental rules. (See Proofs by casting out 11's.)

COROLLARY 2.—In any notation having a regular scale, this property belongs to the number which is one more than the radix. Thus, in the senary notation it belongs to *seven*, in the octary notation it belongs to *nine*, &c.

Proposition XVI.—*Every prime number, that is, every indivisible number, except 2 and 5, ends in 1, or 3, or 7, or 9.*

NOTE.—The converse of this proposition is not true, namely, that every number ending in 1, or 3, or 7, or 9, is a prime number. Thus, 51, 63, 77, and 99 are composite, though they end respectively in 1, 3, 7, and 9.

Art. 101. To find all the prime numbers within any given limit.

CONDITION 1.—Since every even number, except 2, is composite, the other prime numbers must be sought among the odd numbers.

CONDITION 2.—In a series of odd numbers, written in numerical order, thus, 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, &c., every third number after 3 is every number that is divisible by 3, every fifth number after 5 is every number that is divisible by 5, every seventh number after 7 is every number that is divisible by 7, &c.

Rule.—*Write all the odd numbers, within the given limit, in their numerical order. Cancel every third number from 3, every fifth number from 5, every seventh number from 7, and so on. The numbers not cancelled, and the number 2, are all the prime numbers within the given limit.*

ILLUSTRATION.

In the series, 1, 3, 5, 7, ~~9~~, 11, 13, ~~15~~, 17, 19, ~~21~~, 23, ~~25~~, ~~27~~, 29, 31, ~~33~~, ~~35~~, 37, ~~39~~, 41, 43, ~~45~~, 47, ~~49~~, ~~51~~, 53, ~~55~~, ~~57~~, 59, 61, ~~63~~, ~~65~~, 67, ~~69~~, 71, 73, ~~75~~, ~~77~~, 79, ~~81~~, 83, ~~85~~, ~~87~~, 89, ~~91~~, ~~93~~, ~~95~~, 97, ~~99~~, the numbers not cancelled, and the number 2, are all the prime numbers within the limit 100.

NOTE.—It is said that Eratosthenes, a Greek mathematician who was born 275 years before Christ, was the inventor of this method of finding prime numbers, and that the method was called *Eratosthenes' sieve*, because he wrote the series upon parchment, and cut out all the composite numbers, thus giving the parchment the appearance of a sieve. No more expeditious method of finding prime numbers has yet been discovered.

All the prime numbers from 1 to 2207 are given in the table on the following page.

Table of Prime Numbers.

1	137	317	523	743	977	1213	1453	1693	1951
2	139	331	541	751	983	1217	1459	1697	1973
3	149	337	547	757	991	1223	1471	1699	1979
5	151	347	557	761	997	1229	1481	1709	1987
7	157	349	563	769	1009	1231	1483	1721	1993
11	163	353	569	773	1013	1237	1487	1723	1997
13	167	359	571	787	1019	1249	1489	1733	1999
17	173	367	577	797	1021	1259	1493	1741	2003
19	179	373	587	809	1031	1277	1499	1747	2011
23	181	379	593	811	1033	1279	1511	1753	2017
29	191	383	599	821	1039	1283	1523	1759	2027
31	193	389	601	823	1049	1289	1531	1777	2029
37	197	397	607	827	1051	1291	1543	1783	2039
41	199	401	613	829	1061	1297	1549	1787	2053
43	211	409	617	839	1063	1301	1553	1789	2063
47	223	419	619	853	1069	1303	1559	1801	2069
53	227	421	631	857	1087	1307	1567	1811	2081
59	229	431	641	859	1091	1319	1571	1823	2083
61	233	433	643	863	1093	1321	1579	1831	2087
67	239	439	647	877	1097	1327	1583	1847	2089
71	241	443	653	881	1103	1361	1597	1861	2099
73	251	449	659	883	1109	1367	1601	1867	2111
79	257	457	661	887	1117	1373	1607	1871	2113
83	263	461	673	907	1123	1381	1609	1873	2129
89	269	463	677	911	1129	1399	1613	1877	2131
97	271	467	683	919	1151	1409	1619	1879	2137
101	277	479	691	929	1153	1423	1621	1889	2141
103	281	487	701	937	1163	1427	1627	1901	2143
107	283	491	709	941	1171	1429	1637	1907	2153
109	293	499	719	947	1181	1433	1657	1913	2161
113	307	503	727	953	1187	1439	1663	1931	2179
127	311	509	733	967	1193	1447	1667	1933	2203
131	313	521	739	971	1201	1451	1669	1949	2207

Art. 102. To determine whether a number is prime or composite.

Rule.—*If the number is found in the Table of Prime Numbers, it is prime. If it does not exceed 2207, and is not found in the table, it is composite. If it exceeds 2207, and ends in 0, 2, 4, 5, 6, or 8, it is composite. If it exceeds 2207, and ends in 1, 3, 7, or 9, divide it by the successive primes except 1. If an exact divisor is thus found, the number is composite; if an exact divisor is not found, and the quotient is less than the divisor, the number is prime.*

NOTE 1.—When the quotient is less than the divisor, and not exact, it shows, First, that no number *greater* than the trial divisor (except the number itself,) is a divisor, since its quotient, (which would be less than the present trial divisor,) would then be a divisor; and this cannot be, because taking the successive primes as trial divisors has excluded it. Second, that no number *less* than the trial divisor, except 1 is a divisor, since, if prime, it has been tried, and, if composite, its *prime factors* (which would be divisors, because every number is a divisor of all its multiplies,) have been tried.

NOTE 2.—The foregoing rule and table will determine the primes as far as the square of the last number in the table, namely, 2207 × 2207, 4870849.

EXAMPLES FOR PRACTICE.

Are the following numbers prime, or composite?

1. 1567.	5. 9434.	9. 3391.	13. 3407.
2. 1878.	6. 8375.	10. 7091.	14. 11167.
3. 3520.	7. 5466.	11. 3343.	15. 3389.
4. 7582.	8. 4328.	12. 14333.	16. 6749.

FACTORING.

Art. 103. **Factoring** is finding the factors of a number. It is often called *resolving a number into its factors.*

Proposition.—*Every number equals the product of its prime factors.*

DEMONSTRATION.—If all the factors are prime, the proposition is clearly true. If some of the factors are composite, those factors *are produced by* factors either prime or composite. If by com-

posite, the composite factors are again resolvable into factors either prime or composite. Thus, at the last possible resolution, all the factors must be prime numbers.

ILLUSTRATION.

The number,	1152 =	$24 \times 48.$
First resolution,	=	$(3 \times 8) \times (4 \times 12).$
Second "	=	$(3 \times 2 \times 4) \times (2 \times 2 \times 3 \times 4).$
Third "	=	$(3 \times 2 \times 2 \times 2) \times (2 \times 2 \times 3 \times 2 \times 2).$

In which the factors of 1152 are all prime.

COROLLARY.—A number is divisible only by its prime factors, or by some product of them.

Art. 104. To find the prime factors of a number.

Ex. 1. What are the prime factors of 805?

Ans. 5, 7, 23.

WRITTEN PROCESS.

$$\begin{array}{r} 5\,)\,805 \\ \hline 7\,)\,161 \\ \hline 23 \end{array}$$

EXPLANATION.

Since the number ends with 5, we see that 5 is a prime factor of it. (Art. 100, Prop. 12.) Hence dividing by 5, we resolved 805 into 5×161. As 161 ends in 1, it is not divisible by 2, 4, 5, or 6. Trying 3, we find it not a divisor, or factor. Trying 7, we find it is a factor, resolving 161 into 7×23. But 23 is prime. Therefore, the prime factors of 805 are 5, 7, and 23.

Rule.—*Divide the given number by any prime number, except 1, which is contained in it without a remainder. Divide the quotient, if it is composite, in the same manner. Do thus till a quotient is found which is a prime number. The last quotient and all the divisors are the prime factors of the given number.*

PROOF.—Multiply together all the prime factors. The product should be the given number.

NOTE 1.—The judgment of the student must be guided by the facts stated in the propositions given in Art. 100, on divisibility of integers. When the prime factors are small, such as 2, 3, 5, 7, or 11, they can be readily found by the foregoing rule; but, when they are large, the process is often tedious, because of the number of trials which must be made. When the number is even we should divide by 2, so continuing till the quotient is odd. Then we can rapidly try the smaller odd prime numbers.

NOTE 2.—To avoid waste of time in making trials, the student can use a *factor table*. That given on the next two pages contains all the composite numbers less than 10000 which are not divisible by 2, 3, 5, 7, or 11. Their *least prime factors* are placed at their right-hand in small figures. To use the table, proceed by the following:—

Rule.—*First find all the prime factors of the given number which are less than 13. If the last quotient is contained in the table, it is a composite number, and its least prime factor is in small figures at the right. Divide by this prime factor. If the quotient thus obtained is in the table, proceed as before.*

EXAMPLES FOR PRACTICE.

2. Find the prime factors of 9971. Ans. 13, 13, 59.

NOTE.—Finding 9971 in the factor tables, we divide by its least prime factor, 13, there given; then, finding the quotient, 767, in the factor table, we divide by its least prime factor. Since the quotient, 59, is prime, the prime factors of 9971 are 13, 13, and 59.

Resolve into prime factors,

3.	2310.	Ans. 2, 3, 5, 7, 11.	18.	10051.	Ans. 19, 23, 23.
4.	1560.	Ans. 2, 2, 2, 3, 5, 13.	19.	9991.	
5.	2100.		20.	39401.	
6.	1000.		21.	4433.	
7.	2445.		22.	15523.	
8.	6500.		23.	2805.	
9.	3025.		24.	16170.	Ans. 2, 3, 5, 7, 7, 11.
10.	2025.		25.	847.	
11.	1225.		26.	89093.	
12.	441.		27.	7139.	
13.	6561.		28.	5291.	
14.	4536.		29.	2601.	
15.	3645.		30.	4823.	
16.	5082.		31.	4020.	
17.	1734.	Ans. 2, 3, 17, 17.	32.	8652.	

FACTOR TABLE.

No.	Fac.	No.	Fac.	No.	Fac.	No.	Fac.	No.	Fac.	No.	Fac.	No.	Fac.	No.	Fac.	No.	Fac.
169	13	1079	13	1751	17	2353	13	2951	13	3481	59	4061	31	4579	19	5129	23
221	13	1081	23	1763	41	2363	17	2977	13	3497	13	4063	17	4589	13	5141	53
247	13	1121	19	1769	29	2369	23	2983	19	3503	31	4069	13	4601	43	5143	37
289	17	1139	17	1781	13	2407	29	2987	29	3523	13	4087	61	4607	17	5149	19
299	13	1147	31	1807	13	2413	19	2993	41	3551	53	4097	17	4619	31	5161	13
323	17	1157	13	1817	23	2419	41	3007	31	3569	43	4117	23	4633	41	5177	31
361	19	1159	19	1819	17	2449	31	3013	23	3587	17	4121	13	4661	59	5183	71
377	13	1189	29	1829	31	2461	23	3029	13	3589	37	4141	41	4667	13	5191	29
391	17	1207	17	1843	19	2479	37	3043	17	3599	59	4163	23	4681	31	5207	41
403	13	1219	23	1849	43	2483	13	3053	43	3601	13	4171	43	4687	43	5213	13
437	19	1241	17	1853	17	2489	19	3071	37	3611	23	4181	37	4693	13	5219	17
481	13	1247	29	1891	31	2491	47	3077	17	3629	19	4183	47	4699	37	5221	23
493	17	1261	13	1909	23	2501	41	3097	19	3649	41	4187	53	4709	17	5239	13
527	17	1271	31	1919	19	2507	23	3103	29	3653	13	4189	59	4717	53	5249	29
529	23	1273	19	1921	17	2509	13	3107	13	3667	19	4199	13	4727	29	5251	59
533	13	1313	13	1927	41	2533	17	3127	53	3679	13	4223	41	4747	47	5263	19
551	19	1333	31	1937	13	2537	43	3131	31	3683	29	4237	19	4757	67	5267	23
559	13	1339	13	1943	29	2561	13	3133	13	3713	47	4247	31	4769	19	5287	17
589	19	1343	17	1957	19	2567	17	3139	43	3721	61	4267	17	4771	13	5293	67
611	13	1349	19	1961	37	2573	31	3149	47	3737	37	4303	13	4777	17	5311	47
629	17	1357	23	1963	13	2581	29	3151	23	3743	19	4307	59	4811	17	5317	13
667	23	1363	29	2021	43	2587	13	3161	29	3749	23	4309	31	4819	61	5321	17
689	13	1369	37	2033	19	2599	23	3173	19	3757	13	4313	19	4841	47	5329	73
697	17	1387	19	2041	13	2603	19	3193	31	3763	53	4321	29	4843	29	5339	19
703	19	1391	13	2047	23	2623	43	3197	23	3781	19	4331	61	4847	37	5353	53
713	23	1403	23	2059	29	2627	37	3211	13	3791	17	4343	43	4849	13	5359	23
731	17	1411	17	2071	19	2641	19	3233	53	3799	29	4351	19	4853	23	5363	31
767	13	1417	13	2077	31	2669	17	3239	41	3809	13	4369	17	4859	43	5371	41
779	19	1457	31	2117	29	2701	37	3247	17	3811	37	4379	29	4867	31	5377	19
793	13	1469	13	2119	13	2743	13	3263	13	3827	43	4381	13	4883	19	5389	17
799	17	1501	19	2147	19	2747	41	3277	29	3841	23	4387	41	4891	67	5429	61
817	19	1513	17	2159	17	2759	31	3281	17	3859	17	4393	23	4897	59	5447	13
841	29	1517	37	2171	13	2771	17	3287	19	3869	53	4399	53	4901	13	5459	53
851	23	1537	29	2173	41	2773	47	3293	37	3887	13	4427	19	4913	17	5461	43
871	13	1541	23	2183	37	2809	53	3317	31	3893	17	4429	43	4927	13	5473	13
893	19	1577	19	2197	13	2813	29	3337	47	3901	47	4439	23	4979	13	5491	17
899	29	1591	37	2201	31	2831	19	3341	13	3937	31	4453	61	4981	17	5497	23
901	17	1633	23	2209	47	2839	17	3349	17	3953	59	4469	41	4997	19	5513	37
923	13	1643	31	2227	17	2867	47	3379	31	3959	37	4471	17	5017	29	5539	29
943	23	1649	17	2231	23	2869	19	3383	17	3961	17	4489	67	5029	47	5543	23
949	13	1651	13	2249	13	2873	13	3397	43	3973	29	4511	13	5041	71	5549	31
961	31	1679	23	2257	37	2881	43	3401	19	3977	41	4531	23	5053	31	5561	67
989	23	1681	41	2263	31	2899	13	3403	41	3979	23	4537	13	5057	13	5567	19
1003	17	1691	19	2279	43	2911	41	3419	13	3991	13	4541	19	5063	61	5587	37
1007	19	1703	13	2291	29	2921	23	3427	23	4009	19	4553	29	5069	37	5597	29
1027	13	1711	29	2323	23	2923	37	3431	47	4031	29	4559	47	5083	13	5603	13
1037	17	1717	17	2327	13	2929	29	3439	19	4033	37	4573	67	5111	19	5609	71
1073	29	1739	37	2329	17	2941	17	3473	23	4043	13	4577	23	5123	47	5611	31

Factor Table.

No.	Fac.	No.	Fac.	No.	Fac.	No.	Fac.	No.	Fac.	No.	Fac.	No.	Fac.	No.	Fac.	No.	Fac.
5617	41	6119	29	6631	19	7141	37	7627	29	8077	41	8549	83	9071	47	9557	19
5627	17	6137	17	6641	29	7153	23	7631	13	8083	59	8551	17	9073	43	9563	73
5629	13	6157	47	6647	17	7157	17	7633	17	8119	23	8557	43	9077	29	9571	17
5633	43	6161	61	6649	61	7163	13	7657	13	8131	47	8567	13	9083	31	9577	61
5671	53	6169	31	6667	59	7169	67	7661	47	8137	79	8579	23	9089	61	9589	43
5681	13	6179	37	6683	41	7171	71	7663	79	8143	17	8587	31	9101	19	9593	53
5699	41	6187	23	6697	37	7181	43	7697	43	8149	29	8593	13	9113	13	9599	29
5707	13	6191	41	6707	19	7199	23	7709	13	8153	31	8611	79	9131	23	9607	13
5713	29	6227	13	6731	53	7201	19	7729	59	8159	41	8621	37	9139	13	9617	59
5723	59	6233	23	6739	23	7223	31	7739	71	8177	13	8633	89	9143	41	9637	23
5729	17	6239	17	6749	17	7241	13	7747	61	8189	19	8639	53	9167	89	9641	31
5759	13	6241	79	6751	43	7261	53	7751	23	8201	59	8651	41	9169	53	9659	13
5767	73	6253	13	6757	29	7267	13	7769	17	8203	13	8653	17	9179	67	9671	19
5771	29	6283	61	6767	67	7277	19	7771	19	8207	29	8671	13	9193	29	9673	17
5773	23	6289	19	6773	13	7279	29	7781	31	8213	43	8683	19	9197	17	9683	23
5777	53	6313	59	6799	13	7289	37	7783	43	8227	19	8711	31	9211	61	9701	89
5809	37	6319	71	6817	17	7291	23	7787	13	8249	73	8717	23	9217	13	9703	31
5833	19	6331	13	6821	19	7303	67	7801	29	8251	37	8749	13	9223	23	9707	17
5837	13	6341	17	6847	41	7313	71	7807	37	8257	23	8759	19	9253	19	9727	71
5891	43	6371	23	6851	13	7319	13	7811	73	8279	17	8773	31	9259	47	9731	37
5893	71	6383	13	6859	19	7327	17	7813	13	8299	43	8777	67	9263	59	9761	43
5899	17	6401	37	6877	13	7339	41	7831	41	8303	19	8791	59	9269	13	9763	13
5909	19	6403	19	6887	71	7361	17	7837	17	8321	53	8797	19	9271	73	9773	29
5911	23	6407	43	6889	83	7363	37	7849	47	8333	13	8801	13	9287	37	9797	97
5917	61	6409	13	6893	61	7367	53	7859	29	8339	31	8809	23	9299	17	9799	41
5921	31	6431	59	6901	67	7373	73	7871	17	8341	19	8843	37	9301	71	9809	17
5933	17	6437	41	6913	31	7379	47	7891	13	8347	17	8851	53	9307	41	9827	31
5941	13	6439	47	6929	13	7387	83	7897	53	8357	61	8857	17	9313	67	9841	13
5947	19	6443	17	6931	29	7391	19	7913	41	8359	13	8873	19	9322	19	9847	43
5959	59	6463	23	6943	53	7397	13	7921	89	8381	17	8879	13	9347	13	9853	59
5963	67	6467	29	6953	17	7409	31	7939	17	8383	83	8881	83	9353	47	9869	71
5969	47	6487	13	6973	19	7421	41	7943	13	8399	37	8891	17	9367	17	9881	41
5977	43	6493	43	6989	29	7423	13	7957	73	8401	31	8903	29	9379	83	9893	13
5983	31	6497	73	7003	47	7429	17	7961	19	8411	41	8909	59	9389	41	9899	19
5989	53	6499	67	7009	43	7439	43	7967	31	8413	47	8917	37	9407	23	9913	23
5993	13	6509	23	7031	79	7453	29	7969	13	8417	19	8927	79	9409	97	9917	47
6001	17	6511	17	7033	13	7463	17	7979	79	8441	23	8947	23	9451	13	9937	19
6019	13	6527	61	7037	31	7471	31	7981	23	8453	79	8957	13	9469	17	9943	61
6023	19	6533	47	7061	23	7493	59	7991	61	8471	43	8959	17	9481	19	9953	37
6031	37	6539	13	7067	37	7501	13	7999	19	8473	37	8977	47	9487	53	9959	23
6049	23	6541	31	7081	73	7519	73	8003	53	8479	61	8983	13	9503	13	9971	13
6059	73	6557	79	7087	19	7531	17	8021	13	8483	17	8989	89	9509	37	9979	17
6071	13	6583	29	7093	41	7543	19	8023	71	8489	13	8993	17	9517	31	9983	67
6077	59	6593	19	7097	47	7571	67	8027	23	8497	29	9017	71	9523	89	9991	97
6103	17	6613	17	7099	31	7597	71	8033	29	8507	47	9019	29	9529	13	9997	13
6107	31	6617	13	7111	13	7613	23	8047	13	8509	67	9047	83	9553	41	10001	73
6109	41	6623	57	7123	17	7619	19	8051	83	8531	19	9061	13				

Art. 105. To find all the factors, or divisors, of a number.

Proposition.—*All the factors of a number are its prime factors and all possible products of two or more of those factors.*

DEMONSTRATION.—This is a corollary from Proposition XIII, Art. 100.

Ex. 1. What are all the factors, or divisors, of 4725?

WRITTEN PROCESS.

Prime Factors.	Dividends.	Answers.
3	4725	1
3	1575	3
3	525	9
5	175	27
5	35	5, 15, 45, 135.
7	7	25, 75, 225, 675.
	1	7, 21, 63, 189, 35, 105, 315, 945, 175, 525, 1575, and 4725.

EXPLANATION.

By Art. 104, the prime factors of 4725 are, 3, 3, 3, 5, 5, and 7. Since 1 is a divisor, or factor, of every number, we place 1 at the head of the column of answers. To form the other answers, multiply 1 by the next prime factor 3, making 3: then that by the next prime factor 3, making 9: then that by the next 3, making 27. Now multiply by the next prime factor, 5, in the same manner, making 5, 15, 45, and 135. Now multiply all the preceding answers by 5 by the next prime factor 5, making 25, 75, 225, and 675. Now multiply every preceding answer by the last prime factor, 7, making 7, 21, 63, 189, 35, 105, 315, 945, 175, 535, 1575, and 4725. This process manifestly produces all the possible products of all the prime factors, and causes these factors to be represented only once among the answers.

Rule.—*Resolve the number into its prime factors. Write 1 as the first answer. Multiply it by the first prime factor; the product is the next answer. So continue, multiplying by each prime factor all the answers previously found, taking care not to write the same product twice.*

NOTE.—In this case, a number is called a factor, or divisor of itself, and 1 is called a factor or divisor of every number.

Examples for Practice.

Find all the factors or divisors.

1. Of 30. Ans. 1, 2, 3, 5, 6, 10, 15, 30.
2. Of 84. Ans. 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, 84.
3. Of 105. Ans. 1, 3, 5, 7, 15, 21, 35, 105.
4. Of 264. Ans. 1, 2, 3, 4, 6, 8, 11, 12, 22, 24, 33, 44, 66, 88, 132, 264.
5. Of 1470. Ans. 1, 2, 3, 6, 5, 10, 15, 30, 7, 14, 21, 42, 35, 70, 105, 210, 49, 98, 147, 294, 245, 490, 735, 1470.
6. Of 2145. Ans. 1, 3, 5, 15, 11, 33, 55, 165, 13, 39, 65, 195, 143, 429, 715, 2145.
7. Of 2310.

Art. 106. To find how many divisors a number has, without finding every one.

Ex. 1. How many divisors has 4725? Ans. 24.

Analysis.—The prime divisors of 4725 are 3, 3, 3, 5, 5, and 7. Since 3 is a divisor three times its set of answers numbers four, that is 1, 3, 9, 27. Since 5 is a divisor twice, its set of answers numbers three, that is, 1, 5, 25. Since 7 is a divisor once, its set of answers numbers two, that is, 1, 7. Now, the first set, multiplied by the second will give $4 \times 3 = 12$ answers, and each of these 12 being multiplied by the third set, the number of answers resulting is $2 \times 12 = 24$.

Rule.—*Add 1 to the number of times that each prime factor is a divisor, and multiply the results together.*

Examples for Practice.

How many divisors has

2. 30.		6. 455.		10. 630.	
3. 72.		7. 1001.	Ans. 8.	11. 480.	
4. 120.	Ans. 16.	8. 1050.	Ans. 24.	12. 225.	
5. 210.	Ans. 16.	9. 2310.	Ans. 32.	13. 817.	Ans. 4.

Art. 107. When a number is an exact divisor, or measure, of two or more numbers, it is said to be a divisor, or measure, *common to them.*

A **common divisor,** or **common measure,** of two or more numbers is an exact divisor of each of them. Thus, 3 is a common divisor, or common measure, of 6, 9, and 12.

Art. 108. To find the common divisors of two or more whole numbers.

Ex. 1. What are the common divisors of 24, 36, and 48?

ANALYSIS.

By Art. 105, the divisors of 24 are, 1, 2, 3, 4, 6, 8, 12, 24.
" " " " " " 36 " 1, 2, 3, 4, 6, 9, 12, 18, 36.
" " " " " " 48 " 1, 2, 3, 4, 6, 8, 12, 16, 24, 48.

Since each of these sets of divisors contains 1, 2, 3, 4, 6, and 12, these are all the divisors common to 24, 36, and 48. In such cases it is customary to omit the mention of 1, as it is, of course, always a divisor.

Rule.—*Find all the divisors of each number separately. The divisors which are in every set are the common divisors.*

NOTE.—Numbers which are *prime to each other* (See Art. 98,) are not considered as having a common divisor, though strictly speaking, 1 is a divisor common to them.

EXAMPLES FOR PRACTICE.

Find all the common divisors,

2. Of 18, 36, and 54. Ans. 2, 3, 6, 9, 18.
3. Of 12, 24, and 60.
4. Of 32, 48, 96, and 160.
5. Of 70, 105, 175, and 525.
6. Of 90, 120, 150, and 210. Ans. 2, 3, 5.
7. Of 462, 2310, 924, and 3134. Ans. 2, 3, 7, 11.
8. Of 153, 255, 357, and 561.
9. Of 126, 210, 294, and 378.
10. Of 168, 252, and 924.
11. Of 715, 858, and 1144. Ans. 11, 13.
12. Of 231, 616, and 770. Ans. 7, 11.

Art. 109. The **greatest common divisor**, or greatest common measure, of two or more numbers is the greatest number which is an exact divisor of each of them. Thus, in Ex. 1, Art. 108, of the divisors which are common to 24, 36, and 48, the greatest is 12.

Art. 110. To find the greatest common divisor of two or more whole numbers.

Ex. 1. What is the greatest common divisor of 36 and 54?

Ans. 18.

FIRST METHOD.

$36 = 2 \times 2 \times 3 \times 3.$
$54 = 2 \times 3 \times 3 \times 3.$
In both sets are $2 \times 3 \times 3, = 18.$

SECOND METHOD.

$$\begin{array}{r|rr} 9 & 36 & 54 \\ \hline 2 & 4 & 6 \\ \hline & 2 & 3 \end{array}$$

$9 \times 2 = 18$, Ans.

EXPLANATION.—In the first method each number is resolved into its prime factors, and the greatest divisor of each number is plainly the product of all the factors found in both, namely, $2 \times 3 \times 3, = 18$.

In the second method the divisors of both numbers are found, whether prime or not, and the greatest divisor of each number is the product of all the divisors of both, namely, $9 \times 2, = 18$.

Ex. 2. What is the greatest measure common to 390 and 3003?

Ans. 39.

THIRD METHOD.

```
390)3003(7
    2730
    ----
     273)390(1
         273
         ---
         117)273(2
             234
             ---
              39)117(3
```

EXPLANATION.

Since the greatest measure common to 390 and 3003 cannot exceed 390, if 390 measures both, it must be the greatest common measure. But it does not measure 3003, because 273 remains. But *any* number which measures 390 and 3003 must also measure their difference, 273. (Art. 100, Propositions *VI and VII.*) Therefore the *greatest* measure of 390 and 303 must

measure 273, and cannot exceed 273. Now, if 273 measures 390, it must be the greatest measure of 390, and 303, ($= 7 \times 390 + 273$.) But it does not measure 390, because 117 remains. Hence the greatest divisor of 390 and 3003 cannot exceed 117. Now, if 117 measures 273, 117 must be the greatest measure of 390 and 3003.

But it does not measure 273, because 39 remains. Hence the greatest divisor of 390 and 3003 cannot exceed 39. Now, if 39 measures 117, it must be the greatest measure of 390 and 3003. Since 39 does measure 117, 39 is the greatest measure common to 390 and 3003.

Rules. I. *Find the product of all the prime factors common to the given numbers.* Or

II. *Find the product of all the divisors common to the given numbers.* Or

III. *Divide the greater number by the less, and, if there is a remainder, divide the preceding divisor by it. So continue, dividing the last divisor by the last remainder, till nothing remains. The last divisor will be the greatest common divisor of these two numbers.*

When there are more than two numbers, first find the greatest common divisor of two of them, then of that common divisor and one of the other numbers, and so on till every number has been used. The last common divisor is the greatest common divisor of all the numbers.

Examples for Practice.

Find the greatest common divisor of

3. 60, and 270.	Ans. 30.	9. 120, 165, and 210.	Ans. 15.
4. 180, and 300.	Ans. 60.	10. 90, 135, and 198.	Ans. 9.
5. 91, and 143.	Ans. 13.	11. 225, 495, and 360.	
6. 85, and 102.	Ans. 17.	12. 1491, and 1775.	Ans. 71.
7. 341, and 465.		13. 585, and 819.	
8. 299, and 575.		14. 1066, and 2050.	

LEAST COMMON MULTIPLE.

Art. 111. A multiple of a whole number is produced by multiplying that number by a whole number. (Art. 97.)

No number can have a *greatest multiple,* because no multiplier can be found so great that there can be no greater.

A **common multiple** of two or more numbers is a number which is divisible by each of them. Thus, 6 is a multiple which is common to 2 and 3; 12 is a multiple which is common to 2, 3, 4, and 6; and 24 is a common multiple of 2, 3, 4, 6, 8, and 12.

Any set of numbers has an unlimited number of common multiples. Thus, if 6 is a multiple common to 2 and 3, then all the unlimited number of multiples of 6 are also multiples common to 2 and 3. (Art. 100, Prop. III.) Hence no set of numbers can have a greatest common multiple.

The **least common multiple** of two or more numbers is the least number which is divisible by each of them.

Art. 112. The product of one number by another is a common multiple of both numbers, because it is divisible by each of them.

Other common multiples of those numbers may be found by finding the product of this multiple by other whole numbers.

Art. 113. To find the least common multiple of two or more numbers.

The methods of finding the least common multiple of two or more numbers depend on the following

PRINCIPLES.

I. *A multiple of a number must contain all the prime factors of that number.*

II. *A common multiple of two or more numbers must contain all the prime factors of those numbers.*

III. *The least common multiple of two or more numbers must contain each prime factor of those numbers as many times only as are necessary to produce the numbers.*

Ex. 1. What is the least multiple common to 6, 15, and 25?

Ans. 150.

FIRST METHOD.

$$6 = 2 \times 3$$
$$15 = 5 \times 3$$
$$25 = 5 \times 5$$

$5 \times 5 \times 3 \times 2 = 150$, Ans.

EXPLANATION.

Resolving the numbers into their prime factors, we see that the answer must contain 2×3, because it must contain 6. The answer must also contain 5×3, because it must contain 15. But in 2×3 we already have one 3.

Hence we need to add only the 5 to 2×3, making $2 \times 3 \times 5$, to have a set containing all the factors of 6 and 15. But the required set of factors must also contain all the factors of 25, namely, 5×5. Now since one 5 is already in the selected set, we need to add only one more 5, making $2 \times 3 \times 5 \times 5$, to have a set containing all the prime factors of 6, 15, and 25, and only enough to make the component factors of each number. Hence $2 \times 3 \times 5 \times 5, = 150$, is the least common multiple of 6, 15, and 25.

SECOND METHOD.

$$\begin{array}{r|rrr} 3 & 6 & 15 & 25 \\ \hline 5 & 2 & 5 & 25 \\ \hline & 2 & 1 & 5 \end{array}$$

$3 \times 5 \times 2 \times 5 = 150$, Ans.

EXPLANATION.

The first method selected from the sets of the prime factors of the numbers only those which were necessary to produce them. The second method finds those factors by dividing. Since 3 is common to 6 and 15, we select it, leaving 2, 5, and 25. Since 5 is common to 5 and 25, we select it, leaving 2, 1, and 5. Since no factor, except 1, is common to the remaining factors 2, 1, and 5, the factors necessary to produce all the numbers must be 3, 5, 2, and 5, whose product is 150, the least common multiple.

Ex. 2. What is the least common multiple of 14, 28, 35, and 56? Ans. 280.

FIRST METHOD.

$$\begin{array}{l} 7\,)\,\underline{\cancel{14} \quad \cancel{28} \quad 35 \quad 56} \\ \qquad\qquad\qquad\;\; 5 \quad\; 8 \end{array}$$

$7 \times 5 \times 8 = 280$, Ans.

EXPLANATION.

Since 56 contains 14 and 28, a multiple of 56 will contain 14 and 28. (Art. 100, Prop. III.) Hence it is only necessary to obtain the least common multiple of 35 and 56. That is, *when a given number is contained in another given number, it may be neglected in the operation.*

SECOND METHOD.

$$\begin{array}{cccc} \cancel{14} & \cancel{28} & 35 & 56 \\ & & & 8 \end{array}$$

$35 \times 8 = 280$, Ans.

EXPLANATION.

Neglecting 14 and 28 as before, we attend only to 35 and 56. Since the least common multiple must contain 35, the least of these two numbers, we need only to find those factors of 56 which are not factors of 35. This is done by rejecting from 56 any factor also in 35. This is 7, and the other factor of 56 is 8. Hence $8 \times 35, = 280$ is the least common multiple of 14, 28, 35, and 56.

Ex. 3. What is the least common multiple of 15, 24, 40, and 140? Ans. 840.

FIRST METHOD.

4	15	24	40	140
5	15	6	10	35
3	3	6	2	7
2	1	2	2	7
	1	1	1	7

$4 \times 5 \times 3 \times 2 \times 7 = 840$, Ans.

EXPLANATION.

This is solved like the second method of Ex. 1, except that a *composite* divisor 4, is used, instead of dividing by the *prime* 2, and the resulting numbers by 2. *When a composite divisor measures as many of the given numbers as any other divisor measures, it can be used, because it causes the selection of as many factors as any prime divisor would select, and no more.*

NOTE.—On trying the composite divisor 10, it will be seen not to give the least common multiple.

SECOND METHOD.

3)15
$24 \times 5 = 120$,
l. c. m. of 15 and 24.

The greatest factor of 15 not in 24 is 5. Hence 24×5, or 120, is the least common multiple of 15 and 24.

40)120
$40 \times 3 = 120$,
l. c. m. of 15, 24, and 40.

The factors of 40 are in 120. Hence 120 is the least common multiple of 15, 24, and 40.

20)120
$140 \times 6 = 840$.
l. c. m. of 15, 24, 40, and 140.

The greatest factor of 120 not in 140 is 6. Hence 140×6, or 840, is the least common multiple of 15, 24, 40, and 140.

EXPLANATION.

Rules. I. *If the numbers are prime to each other, find their product.*

II. *Resolve the numbers into their prime factors, and multiply together all those of the largest number and such of the others as are not in the largest number. The product is the least common multiple.*

III. *Write the numbers in a line, and divide by any prime divisor of two or more of them. Write the quotients and undivided numbers in a line below. Divide these in the same manner. Continue thus till no number greater than 1 is a divisor of any two of the numbers. The product of the divisors and remaining numbers is the least common multiple.*

IV. *Reserve one of the numbers, and reject from the others the greatest factor common to them and the one reserved. If any of the resulting numbers have common factors, reserve one of the numbers, and reject from the others in like manner. So continue till the resulting numbers are prime to each other. The product of the reserved and resulting number is the least common multiple.*

NOTE 1.—A composite divisor may be used in Rule III, when it is contained in as many of the numbers as any other divisor is.

NOTE 2.—When one of the numbers is a divisor of another, it may be left out of the operation.

EXAMPLES FOR PRACTICE.

4. Find the least common multiple of 14, 35, 56, and 70. Ans. 280.

5. Of 18, 24, 40, and 50.
6. Of 25, 30, 45, 80, and 120.
7. Of 72, 120, 135, and 144.
8. Of 3, 5, 7, 9, 10, 12, and 15.
9. Of 65, 78, 104, and 130. Ans. 1560.
10. Of 35, 50, 75, 84, and 120. Ans. 4200.
11. Of 48, 120, 180 and 288. Ans. 1440.
12. Of 36, 45, 90, 210, and 320.
13. Of 11, 13, 29, and 31.
14. Of 23, 33, and 55.
15. Of 26, 39, 85, and 117.

Art. 114. Proposition.—*The product of the greatest common divisor and least common multiple of two numbers equals the product of the numbers.*

DEMONSTRATION.—Since the greatest common divisor of two numbers contains only the sets of factors common to them, and their least common multiple contains only the sets of factors not common to them, the product of their greatest common divisor and least common multiple must contain the product of all the factors of the numbers, and must therefore equal the product of the numbers. Thus, of 36 and 54, the set of common factors is $2 \times 3 \times 3 = 18$; the sets not common are 2×2 and $3 \times 3 \times 3$, (Art. 113, Rule II.) and $36 \times 54 = (2 \times 3 \times 3) \times (2 \times 2) \times (3 \times 3 \times 3)$, $= 1944$.

COROLLARY 1.—The product of two numbers, divided by their greatest common divisor, equals their least common multiple.

COROLLARY 2.—The least common multiple of two numbers, divided by their greatest common divisor, or factor, equals the product of the other factors of the numbers. From this quotient the other actual factors of the number may be found by resolving the quotient into its prime factors, and combining them.

Art. 115. To find the number which have a given greatest common divisor and a given least common multiple.

Rule.—*Divide the least common multiple by the greatest common divisor. Resolve the quotient into factors which are prime to each other, and multiply the results by the greatest common divisor.*

EXAMPLES FOR PRACTICE.

1. What numbers have for a greatest common divisor 18, and for a least common multiple 108?

Ans. (1). 18, 36, and 54.
" (2). 36, and 54.

Solution.—18) 108 (6. $\left\{ \begin{matrix} =1\times2\times3 \\ = \quad 2\times3 \end{matrix} \right\} \begin{matrix} \times18, \text{ forms } 18, 36, \text{ and } 54, \text{ Set } 1. \\ \times18, \quad \text{"} \quad\quad 36 \text{ and } 54, \text{ Set } 2. \end{matrix}$

2. Of what numbers is 24 the greatest common divisor, and 360 the least common multiple? Ans. 24, 72, and 120.

3. Of what numbers is 21 the greatest common divisor, and 126 the least common multiple?

4. Of what numbers is 35 the greatest common divisor, and 735 the least common multiple?

5. What two numbers have for a greatest common divisor 13, and for a least common multiple 364? Ans. 52, and 91.

6. What two numbers have for a greatest common divisor 17, and for a least common multiple 255? Ans. 51, and 83

7. What three numbers have for a greatest common divisor 18, and for a least common multiple 1890?

Ans. 54, 90, and 126.

SYNOPSIS OF THE PROPERTIES AND RELATIONS OF INTEGERS.

CLASSES.	KINDS.	PROPERTIES AS TO.	RELATIONS AS TO.	OPERATIONS.
Factors, or Divisors, or Measures.	Prime.	Scale structure.	Sum.	Determining the properties
Multiples.	Composite.	Absolute divisibility.	Difference.	Factoring.
	Odd.	Relative divisibility.	Product.	Finding common factors, or divisors.
	Even.		Quotient.	Finding common multiples.
	Perfect.			
	Imperfect.			
	Abundant.			
	Defective.			

CHAPTER IX.

COMMON FRACTIONS.

Art. 116. A **fraction** is a number expressing one or more of the equal parts of a unit.

In reference to the number of parts into which the unit is divided, and to the style of writing them, fractions are of two kinds, namely *common*, and *decimal*.

A **common fraction** is used to express any part or parts of a unit, such as *halves, thirds, fourths, tenths, hundredths,* &c.

A **decimal fraction** is used to express decimal parts of a unit, such as *tenths, hundredths, thousandths,* &c.

NOTE.—Common fractions are often called *vulgar fractions.*

NOTATION OF COMMON FRACTIONS.

Art. 117. A common fraction is expressed by writing one number above another with a line between them. Thus, *two-fifths* is written $\frac{2}{5}$, *three-eighths* $\frac{3}{8}$, *six-tenths* $\frac{6}{10}$, &c.

The **denominator** of a fraction is the number below the line. It is so called because it *denominates*, or names, the parts of the unit. Thus, in $\frac{2}{5}$ the lower number, 5, represents the division of the unit into five equal parts, and the fraction expresses *fifths.*

The **numerator** of a fraction is the number above the line. It is so called because it *numerates*, or numbers, the parts expressed by the fraction. Thus, in $\frac{2}{5}$ the upper number, 2, denotes that two of the five-fifths of a unit are expressed by the fraction. Hence the numerator is a cardinal

number, and the denominator an ordinal adjective, viewed either as a noun, or as having the noun *part* or *parts* understood.

The **terms** of a fraction are its numerator and denominator. They are so named when alluded to in general language, such as, *the upper term*, *the lower term*, *both terms*, &c.

Art. 118. To write a common fraction.

Rule.—*Write the numerator; under it draw a horizontal line, and under the line write the denominator.*

NOTE.—In business, the line between the terms of a fraction is sometimes drawn downwards obliquely to the left, thus, 3/4, &c.

EXAMPLES FOR PRACTICE.

Write in figures,

1. Three-eighths; three-sixteenths; three-twelfths.
2. Five-ninths; five-thirteenths; five-sevenths.
3. Six-elevenths; six-twenty-fifths.
4. Seven-twelfths; eight-forty-firsts.
5. Ten-seventeenths; nine-fortieths.
6. Three-fourths; four-fifths; six-sevenths.
7. Three-elevenths; seven-sixteenths.
8. Thirteen-fifty-firsts; twenty-sixty-firsts.
9. Twenty-five-sixty-thirds; sixty-ninetieths.
10. Seventy-one-one-hundred-and-seconds.
11. One-hundred-and-five-two-hundred-and-thirds.

NUMERATION OF COMMON FRACTIONS.

Art. 119. To read a common fraction.

Rule.—*Pronounce the numerator as a whole number, and then the denominator as an ordinal number.*

NOTE.—When the denominator is 2, it is read as *half* or *halves*, and, when it is 4, it is read as *quarters*, or *fourths*.

EXAMPLES FOR PRACTICE.

Read the fraction formed.

1. By 5 over 21. Ans. Five-twenty-firsts.
2. By 7 over 32. Ans. Seven-thirty-seconds.
3. By 9 over 103. Ans. Nine-one-hundred-and-thirds.
4. By 8 over 1005. Ans. Eight-one-thousand-and-fifths.
5. Read $\frac{1}{2}$: $\frac{3}{2}$: $\frac{5}{12}$: $\frac{13}{22}$: $\frac{79}{102}$: $\frac{681}{2002}$: $\frac{1}{3}$: $\frac{2}{3}$: $\frac{7}{3}$: $\frac{12}{13}$: $\frac{19}{23}$: $\frac{90}{103}$: $\frac{856}{1003}$.
6. Read $\frac{1}{4}$: $\frac{9}{4}$: $\frac{5}{14}$: $\frac{7}{24}$: $\frac{43}{124}$: $\frac{471}{4004}$: $\frac{1}{5}$: $\frac{4}{5}$: $\frac{19}{5}$: $\frac{17}{25}$: $\frac{37}{45}$: $\frac{38}{105}$: $\frac{451}{1005}$.
7. Read $\frac{1}{6}$: $\frac{5}{6}$: $\frac{7}{16}$: $\frac{13}{36}$: $\frac{55}{106}$: $\frac{329}{2306}$: $\frac{1}{7}$: $\frac{6}{7}$: $\frac{16}{7}$: $\frac{16}{17}$: $\frac{16}{37}$: $\frac{16}{107}$: $\frac{16}{1007}$.
8. Read $\frac{1}{8}$: $\frac{7}{8}$: $\frac{11}{18}$: $\frac{31}{48}$: $\frac{73}{108}$: $\frac{4213}{7508}$: $\frac{1}{9}$: $\frac{8}{9}$: $\frac{27}{29}$: $\frac{37}{49}$: $\frac{89}{99}$: $\frac{107}{109}$: $\frac{1005}{1009}$.
9. Read $\frac{1}{10}$: $\frac{9}{10}$: $\frac{11}{100}$: $\frac{59}{90}$: $\frac{83}{400}$: $\frac{17}{5000}$: $\frac{1}{11}$: $\frac{7}{11}$: $\frac{17}{21}$: $\frac{21}{81}$: $\frac{83}{91}$: $\frac{68}{101}$: $\frac{166}{2301}$.
10. Read $\frac{1}{12}$: $\frac{7}{12}$: $\frac{5}{14}$: $\frac{11}{15}$: $\frac{103}{222}$: $\frac{401}{2202}$: $\frac{1}{13}$: $\frac{12}{13}$: $\frac{28}{33}$: $\frac{65}{83}$: $\frac{71}{93}$: $\frac{94}{403}$: $\frac{888}{3333}$.

SIGNIFICATION OF FRACTIONS.

Art. 120. A common fraction may have two meanings, viz.:—

FIRST.—*The division of a unit into equal parts, and the expression of a certain number of these parts.* Thus, 2 over 5, or $\frac{2}{5}$, may represent the division of a unit into 5 equal parts, called *fifths*, and may express two of these fifths, that is, *two-fifths of 1.*

SECOND.—*The division of the numerator into as many equal parts as there are units in the denominator, and the expression of one of those parts.* Thus, $\frac{2}{5}$ may represent the division of 2 into 5 equal parts, and may express one of those parts, that is *one-fifth of 2.*

VALUE OF FRACTIONS.

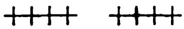
Art. 121. The **value** of a fraction is the quantity which it expresses, and is the quotient of the numerator divided by *the denominator.* Thus, the value of $\frac{6}{3}$ is 2.

DEMONSTRATION.—Since 3 thirds equal 1 unit, 6 thirds are as many units as 3 thirds are contained times in 6 thirds. Since 3 thirds are contained in 6 thirds 2 times, 6 thirds are equal to 2 units.

Both meanings of a fraction have the same value. Thus, *two-fifths of 1* is the same value as *one-fifth of 2*. This appears from the following demonstration and illustration:—

DEMONSTRATION.—Since one is 5 fifths of 1, 2 if equal to 2 times 5 fifths of 1, that is 10 fifths of 1, and 1 fifth of 2 must equal 1 fifth of 10 fifths of 1, that is, 2 fifths of 1.

ILLUSTRATION.



If two equal lines are each divided into 5 equal pieces, 2 of these pieces are two-fifths of one line, and, also, one-fifth of the whole ten pieces, into which the two lines were cut.

The value of a fraction does not depend upon the value of the terms, but upon the number of times that the lower term is contained in the upper. Thus, $\frac{3}{1}$, $\frac{6}{2}$, $\frac{9}{3}$, $\frac{12}{4}$, $\frac{15}{5}$, $\frac{18}{6}$, &c. have the same value, though the terms have different values.

VARIATIONS OF THE VALUES OF FRACTIONS.

Art. 122. Since the value of a fraction is the quotient of the numerator divided by the denominator, it varies with the *relative* variations of the terms. (See Art. 79.)

Preposition I.—*The value of a fraction varies directly as the numerator.*

DEMONSTRATION.—The same denominator is contained in twice the numerator, twice as many times; in three times the numerator, three times as many times; in one-half the numerator, one-half as many times; and one-third of the numerator, one-third as many times, &c.

ILLUSTRATION.—If we double the numerator in $\frac{24}{4} = 6$, we obtain $\frac{48}{4} =$ twice 6, or 12. If we divide the numerator by 2, we obtain $\frac{12}{4} =$ one-half of 6, or 3; &c. Hence, multiplying the numerator by a number multiplies the value of the fraction by that number; and dividing the numerator by a number divides the value of the fraction by that number.

Preposition II.—*The value of a fraction varies inversely as the denominator.*

DEMONSTRATION.—If the numerator is constant, twice the denominator is contained in it one-half as many times; three times

the denominator, one-third as many times; one-half of the denominator, twice as many times; one-third of the denominator, three times as many times, &c.

ILLUSTRATION.—If we double the denominator in $\frac{24}{4} = 6$, we obtain $\frac{24}{8} =$ one half of 6, or 3. If we divide the denominator by 2, we obtain $\frac{24}{2} =$ twice 6, or twelve. Hence, multiplying the denominator of a fraction by a number divides the value of the fraction by that number; and dividing the denominator of a fraction by a number multiplies the value of the fraction by that number.

Preposition III.—*The value of a fraction is constant when the terms vary proportionally.*

DEMONSTRATION.—Twice the denominator is contained in twice the numerator the same number of times that the denominator is contained in the numerator. The same is true of any number of times the denominator in the same number of times the numerator and of any part of the denominator in the same part of the numerator.

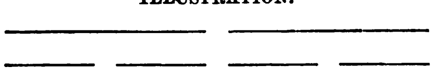
ILLUSTRATIONS.—$\frac{24}{6} = 4; = \frac{48}{12}; = \frac{72}{18}$, &c. $\frac{24}{6} = 4; = \frac{12}{3}; = \frac{8}{2}$, &c. Hence, multiplying or dividing both terms of a fraction by the same number, does not affect the value of the fraction.

EXERCISES.

1. Illustrate the first preposition with $\frac{60}{12} : \frac{210}{105}$.
2. Illustrate the second preposition with $\frac{120}{12} : \frac{630}{105}$.
3. Illustrate the third preposition with $\frac{72}{12} : \frac{120}{24}$.

Art. 123. *The value of one of the equal parts of a unit varies inversely as the number of the parts into which the unit is divided.* Thus, if the number of the equal parts is doubled, each part has only half the value of a former part; if there are three times as many parts, each part has only one-third of the value of a former part; if there are half as many parts, each part has twice the value of a former part, &c.

ILLUSTRATION.



If a line is divided into two equal parts, each part is one-half of the line. But if each half is divided into two equal parts, each of these parts is one-fourth of the whole line, and is also, one-half of one-half of the line. Hence, one-fourth is one-half of the value of one-half.

CLASSIFICATION OF COMMON FRACTIONS.

Art. 124. In reference to their value, common fractions are of two kinds, namely, *proper* and *improper*.

A **proper fraction** is a fraction, whose numerator is less than its denominator; for example, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{9}$, &c. Such a fraction is called *proper*, because its value is less than a unit, and is, therefore, properly expressed in the form of a fraction.

An **improper fraction** is a fraction whose numerator equals or exceeds its denominator; for example, $\frac{3}{3}$, $\frac{6}{2}$, $\frac{8}{5}$, $\frac{15}{7}$, &c. Such a fraction is called *improper*, because its value equals or exceeds a unit, and, therefore, the proper expression for its value is a whole or mixed number. Thus, $\frac{12}{3}$ is properly 4.

Art. 125. In reference to the form of their terms, common fractions are of two kinds, namely, *simple* and *complex*.

A **simple fraction** is a fraction whose terms are whole numbers; for example, $\frac{5}{7}$, $\frac{11}{8}$, $\frac{4}{13}$, $\frac{21}{9}$, &c.

A **complex fraction** is a fraction which has a fraction in one or both of its terms; For example, $\dfrac{\frac{3}{4}}{7}$, $\dfrac{5}{\frac{1}{2}}$, $\dfrac{6\frac{2}{3}}{\frac{4}{5}}$, $\dfrac{\frac{5}{9}}{4\frac{3}{8}}$, $\dfrac{8}{3\frac{2}{7}}$, &c.

NOTE.—As the reading of complex fractions in the usual way is sometimes awkward and inconvenient, it is customary to read them by stating the numerator as *over*, or *divided by* the denominator. Thus, we can read the fractions here given as $\frac{3}{4}$ *over* 7, or $\frac{3}{4}$ *divided by* 7, &c. Some prefer to say *numerator* $\frac{3}{4}$, *denominator* 7.

EXERCISES.

Read the following complex fractions:—

$$\frac{\frac{3}{5}}{8};\ \frac{6}{\frac{3}{4}};\ \frac{\frac{5}{6}}{\frac{7}{8}};\ \frac{\frac{5}{9}}{14};\ \frac{14}{\frac{5}{9}};\ \frac{\frac{5}{9}}{\frac{7}{11}};\ \frac{\frac{7}{16}}{100};\ \frac{\frac{5}{11}}{23};\ \frac{5}{\frac{11}{23}};\ \frac{\frac{11}{13}}{\frac{16}{21}};\ \frac{16}{2\frac{1}{2}};\ \frac{\frac{3}{7}}{2\frac{1}{3}}.$$

Art. 126. In reference to the number of their terms, common fractions are of two kinds, namely, *single* and *compound*.

A **single fraction** is a fraction which has but one numerator and one denominator; for example, $\frac{3}{4}$.

A **compound fraction** is a fraction of a fraction. It is an expression composed of two or more single fractions connected by the word *of*, or by the sign of multiplication; for example, $\frac{1}{2}$ *of* $\frac{4}{5}$; $\frac{1}{2} \times \frac{4}{5}$; $\frac{2}{3}$ *of* $\frac{5}{8}$ *of* $\frac{3}{5}$; $\frac{2}{3} \times \frac{5}{8} \times \frac{3}{5}$.

Art. 127. A **mixed number** is a number composed of an integer and a fraction; for example, $7\frac{5}{9}$; $18\frac{3}{4}$; $16\frac{2}{3}$; &c.

The **reciprocal** of a number is the quotient arising from dividing 1 by that number. Thus, the reciprocal of 5 is $\frac{1}{5}$.

Exercises on the Definitions.

1. Is $\frac{4}{5}$ a proper, or an improper fraction?

Model of Answer.—*Four-fifths* is a proper fraction, because its numerator is less than its denominator, and therefore its value is less than a unit.

$\frac{3}{4}$? $\frac{4}{3}$? $\frac{5}{8}$? $\frac{8}{5}$? $\frac{5}{5}$? $\frac{5}{10}$? $\frac{10}{5}$? $\frac{6}{6}$? $\frac{6}{12}$? $\frac{12}{6}$? $\frac{7}{7}$? $\frac{5}{13}$? $\frac{18}{9}$?

2. Is $\frac{4}{5}$ a simple, or a complex fraction?

Model of Answer.—*Four-fifths* is a simple fraction, because its terms are whole numbers.

$\frac{5}{8}$? $\frac{6}{7}$? $\dfrac{\frac{3}{4}}{5}$? $\frac{5}{13}$? $\dfrac{3}{\frac{4}{5}}$? $\frac{9}{10}$? $\dfrac{\frac{3}{4}}{\frac{5}{6}}$? $\frac{3}{25}$? $\dfrac{2\frac{1}{2}}{3}$? $\dfrac{5}{6\frac{2}{3}}$? $\dfrac{1\frac{3}{4}}{2\frac{2}{5}}$?

3. Is $\frac{4}{5}$ a single, or a compound fraction?

Model of Answer.—*Four-fifths* is a single fraction, because it has but one numerator and one denominator.

$\frac{5}{6}$? $\frac{3}{8}$? $\frac{3}{4}$ of $\frac{5}{6}$? $\frac{2}{3} \times \frac{3}{4}$? $\frac{5}{9}$? $\frac{1}{3}$ of $\frac{3}{5}$? $\frac{7}{12}$? $\frac{5}{6} \times \frac{7}{8}$? $\frac{7}{24}$? $\frac{8}{41}$?

4. What is the number $3\frac{4}{5}$?

Model of Answer.—*Three and four-fifths* is a mixed number, because it is composed of the integer 3 and the fraction $\frac{4}{5}$.

$2\frac{1}{3}$? $5\frac{3}{4}$? $6\frac{3}{5}$? $7\frac{3}{8}$? $8\frac{3}{7}$? $9\frac{5}{9}$? $5\frac{8}{25}$? $12\frac{3}{31}$? $16\frac{2}{3}$? $18\frac{3}{4}$?

Reduction of Common Fractions.

Art. 128. Reduction of fractions is changing them into other equivalent expressions. It also includes the changing of whole and mixed numbers to the form of a fraction.

A fraction is said to be *reduced to higher terms* when its terms are changed to larger numbers; it is said to be *reduced to lower terms* when its terms are changed to smaller numbers.

A fraction is *in it lowest terms* when its terms are prime to each other. (See Art. 98.)

Case I.

Art. 129. To reduce a fraction to proposed higher terms.

Ex. 1. Reduce $\frac{5}{8}$ to 24ths. Ans. $\frac{15}{24}$.

Written Process.

$$\frac{8\,)\,24}{3};\quad \frac{3\times 5}{3\times 8}=\frac{15}{24}.$$

Explanation.

First.—In 1 eighth there are as many 24ths as 8 is contained times in 24, (See Art. 123,) that is, 3 twenty-fourths, and in 5 eighths there are 5 times 3 twenty-fourths, that is, fifteen twenty-fourths.

Second.—If the denominator 8 is made 3 times 8, or 24, the numerator 5 must be made 3 times 5, or 15, to vary the terms proportionally. (See Art. 122.) The fraction $\frac{5}{8}$ thus becomes $\frac{15}{24}$.

Ex. 2. Reduce $\frac{5}{8}$ to a fraction whose numerator is 10.

Ans. $\frac{10}{16}$.

Written Process.

$$\frac{5\,)\,10}{2};\quad \frac{2\times 5}{2\times 8}=\frac{10}{16}.$$

Explanation.

Since 10 is 2 times 5, if the numerator is made 2 times 5, or 10, the denominator 8 must be made 2 times 8, or 16, to vary the terms proportionally. This makes $\frac{10}{16}$.

Rule.—*Divide the proposed term by the given term, and multiply both terms of the fraction by the quotient.*

Note.—According to Preposition III, Art. 122, this reduction does not change the value of the fraction, because it varies its terms proportionally.

EXAMPLES FOR PRACTICE.

Reduce,		to		Reduce,		to numerator.	
3.	$\frac{5}{6}$,	12ths.	Ans. $\frac{10}{12}$.	11.	$\frac{5}{6}$,	30.	Ans. $\frac{30}{36}$.
4.	$\frac{2}{3}$,	18ths.	Ans. $\frac{12}{18}$.	12.	$\frac{7}{8}$,	21.	Ans. $\frac{21}{24}$.
5.	$\frac{3}{4}$,	20ths.		13.	$\frac{3}{4}$,	36.	
6.	$\frac{3}{5}$,	50ths.		14.	$\frac{6}{7}$,	36.	
7.	$\frac{5}{7}$,	49ths.		15.	$\frac{8}{9}$,	80.	
8.	$\frac{4}{5}$,	80ths.		16.	$\frac{13}{25}$,	91.	
9.	$\frac{4}{9}$,	81sts.		17.	$\frac{16}{21}$,	96.	
10.	$\frac{7}{23}$,	92ds.		18.	$\frac{17}{31}$,	119.	

CASE II.

Art. 130. To reduce a fraction to proposed lower terms.

Ex. 1. Reduce $\frac{36}{60}$ to fifteenths. Ans. $\frac{9}{15}$.

WRITTEN PROCESS.

$$15\,)\,\frac{60}{4}\,; \quad \frac{36 \div 4}{60 \div 4} = \frac{9}{15}$$

EXPLANATION.

Since there are one-fourth as many fifteenths as sixtieths in a unit, we reduce the denominator to fifteenths by dividing by 4. To vary the terms proportionally, (Prop. III, Art. 122,) we must divide the numerator 36 by 4, changing the fraction to $\frac{9}{15}$.

Ex. 2. Thirty-six sixtieths are 9 what? Ans. $\frac{9}{15}$.

WRITTEN PROCESS.

$$9\,)\,\frac{36}{4} \quad \frac{36 \div 4}{60 \div 4} = \frac{9}{15}.$$

EXPLANATION.

We find what divisor reduces the numerator 36 to 9 by dividing 36 by 9. This gives the divisor 4. Since we divide the numerator by 4, we must divide the denominator 60 by 4, to vary the terms proportionally. This changes the fraction to $\frac{9}{15}$.

Rule.—*Divide the given term by the proposed term, and divide both terms of the fraction by this quotient.*

NOTE—This reduction does not change the value of the fraction, because it varies the terms proportionally. (Art. 123, Prop. III.)

Examples for Practice.

Reduce,	to		Reduce,	to numerator.	
3. $\frac{28}{36}$,	9ths.	Ans. $\frac{7}{9}$.	8. $\frac{9}{12}$,	3.	Ans. $\frac{3}{4}$.
4. $\frac{28}{49}$,	7ths.	Ans. $\frac{4}{7}$.	9. $\frac{35}{49}$,	5.	
5. $\frac{50}{110}$,	11ths.		10. $\frac{40}{50}$,	4.	
6. $\frac{32}{56}$,	7ths.		11. $\frac{88}{121}$,	8.	
7. $\frac{104}{200}$,	25ths.		12. $\frac{156}{169}$,	12.	

Case III.

Art. 131. To reduce a fraction to its lowest terms.

Ex. 1. Reduce $\frac{36}{60}$ to its lowest terms. Ans. $\frac{3}{5}$.

First Process.

$$\frac{36}{60} = \frac{\cancel{2} \times \cancel{2} \times \cancel{3} \times 3}{\cancel{2} \times \cancel{2} \times \cancel{3} \times 5} = \frac{3}{5}.$$

Second Process.

$$2)\frac{36}{60} = \frac{18}{30}:\ 2)\frac{18}{30} = \frac{9}{15}:\ 3)\frac{9}{15} = \frac{3}{5}.$$

Third Process.

$$12)\frac{36}{60} = \frac{3}{5}.$$

Explanation.

In the first process we resolve both terms into their prime factors, and cancel out all common factors, leaving the terms 3 and 5 relatively prime.

In the second process we cancel common prime factors by a *series* of divisions.

In the third we cancel all common factors by dividing by the greatest common divisior.

Rule.—*Cancel all factors common to the terms.* Or, *Divide both terms by their greatest common divisor.*

Examples for Practice.

Reduce to its lowest terms

2. $\frac{12}{18}$.	Ans. $\frac{2}{3}$.	6. $\frac{25}{40}$.		10. $\frac{54}{81}$.
3. $\frac{36}{48}$.		7. $\frac{24}{32}$.		11. $\frac{52}{84}$.
4. $\frac{45}{72}$.		8. $\frac{35}{56}$.	Ans. $\frac{5}{8}$.	12. $\frac{125}{175}$.
5. $\frac{63}{108}$.		9. $\frac{30}{50}$.		13. $\frac{240}{400}$.

14. $\frac{989}{1127}$, (By factor table,) Ans. $\frac{43}{49}$.
15. $\frac{1421}{1911}$, " Ans. $\frac{29}{39}$.
16. $\frac{2091}{2501}$, " Ans. $\frac{51}{61}$.
17. $\frac{903}{2279}$, Ans. $\frac{21}{53}$.
18. $\frac{1739}{2479}$, Ans. $\frac{47}{67}$.
19. $\frac{4757}{5963}$, Ans. $\frac{71}{89}$.

Case IV.

Art. 132. To find the part that one abstract number is of another.

Ex. 1. What part of 24 is 18? Ans. $\frac{3}{4}$.

WRITTEN PROCESS.

$$6\ )\ \frac{18}{24} = \frac{3}{4}$$

EXPLANATION.

Since 1 unit is one-twenty-fourth of 24 units, eighteen units must be 18 times as much, that is, eighteen-twenty-fourths of 24. Since $\frac{18}{24}$ is, in its lowest terms, $\frac{3}{4}$, 18 is $\frac{3}{4}$ of 24.

Rule.—*Write that number which is considered the part, for a numerator, and the other number for a denominator; then reduce this fraction to its lowest terms.*

Examples for Practice.

What part
2. Of 36 is 27? Ans. $\frac{3}{4}$.
3. Of 50 is 30?
4. Of 30 is 18?
5. Of 32 is 20?

What part
6. Of 80 is 64? Ans. $\frac{4}{5}$.
7. Of 72 is 60?
8. Of 325 is 175?
9. Of 500 is 375?

Case V.

Art. 133. To reduce an improper fraction to its proper terms.

Ex. 1. What is the proper expression for $\frac{173}{12}$. Ans. $14\frac{5}{12}$.

WRITTEN PROCESS.

$$12\ )\ 173$$
$$\overline{14\tfrac{5}{12}}$$

EXPLANATION.

Since $\frac{12}{12}$ are 1 unit, $\frac{173}{12}$ are as many units as 12 is contained times in 173, that is, 14 $\frac{5}{12}$ units.

Rule.—*Divide the numerator by the denominator.*

EXAMPLES FOR PRACTICE.

What is the value

2. Of $\frac{31}{4}$? Ans. $7\frac{3}{4}$.	6. Of $\frac{15}{1}$?	10. Of $\frac{121}{11}$?
3. Of $\frac{36}{6}$?	7. Of $\frac{96}{9}$? Ans. $10\frac{2}{3}$.	11. Of $\frac{7}{7}$?
4. Of $\frac{25}{3}$?	8. Of $\frac{57}{2}$?	12. Of $\frac{182}{5}$?
5. Of $\frac{32}{5}$?	9. Of $\frac{65}{7}$?	13. Of $\frac{324}{16}$?

CASE VI.

Art. 134. To reduce a whole number to a fraction having a given denominator.

Ex. 1. Reduce 12 to sevenths. Ans. $\frac{84}{7}$.

WRITTEN PROCESS.	EXPLANATION.
1 2 7 sevenths in 1. ——— 8 4 sevenths in 12, $= \frac{84}{7}$.	Since 1 unit is 7 sevenths, 12 units are 12 times 7 sevenths, that is 84 sevenths.

Rule.—*Multiply the given number by the proposed denominator, and place the product over that denominator.*

EXAMPLES FOR PRACTICE.

Reduce	Reduce
2. 15 to halves. Ans. $\frac{30}{2}$.	10. 45 to tenths.
3. 16 to thirds.	11. 32 to elevenths.
4. 14 to fourths.	12. 21 to twenty-firsts.
5. 12 to fifths.	13. 22 to seventeenths.
6. 17 to sixths.	14. 43 to thirty-thirds.
7. 19 to sevenths.	15. 32 to forty-thirds.
8. 21 to eighths.	16. 102 to ninths.
9. 23 to ninths.	17. 75 to four-hundredths.

18. Reduce 13 to the form of a fraction. Ans. $\frac{13}{1}$.

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CASE VII.

Art. 135. To reduce a mixed number to an improper fraction.

Ex. 1. Reduce $8\frac{4}{5}$ to an improper fraction. Ans. $\frac{44}{5}$.

WRITTEN PROCESS.

5 fifths in 1.
8
——
4 0 fifths in 8.
4 fifths added in.
——
4 4 fifths in $8\frac{4}{5}$.

EXPLANATION.

Since 1 unit is 5 fifths, 8 units are 8 times 5 fifths, that is, 40 fifths: 40 fifths and 4 fifths are 44 fifths. Therefore, $8\frac{4}{5}$ are equal to $\frac{44}{5}$.

Rule.—*Multiply the whole number by the denominator of the fractional part; to the product add the numerator, and write the sum over the denominator.*

NOTE.—The improper fraction may afterward be changed into a fraction of some other proposed denominator by Articles 129 and 130.

EXAMPLES FOR PRACTICE.

Reduce to an improper fraction.

2. $3\frac{3}{4}$. Ans. $\frac{15}{4}$.	5. $9\frac{3}{5}$.	8. $37\frac{5}{7}$.
3. $4\frac{2}{3}$.	6. $18\frac{3}{4}$.	9. $136\frac{5}{12}$.
4. $5\frac{4}{5}$.	7. $33\frac{1}{8}$.	10. $200\frac{1}{200}$.

11. Reduce $9\frac{7}{8}$ to twenty-fourths.
12. Reduce $20\frac{3}{5}$ to twentieths.

CASE VIII.

Art. 136. To find the value of a compound fraction.

Ex. 1. What is the value of $\frac{3}{5}$ of $\frac{4}{7}$? Ans. $\frac{12}{35}$.

WRITTEN PROCESS.

$$\frac{3}{5} \times \frac{4}{7} = \frac{3 \times 4}{5 \times 7}, = \frac{12}{35}.$$

EXPLANATION.

One-fifth of $\frac{4}{7}$ is $\frac{4}{7} \div 5$, which is $\frac{4}{35}$, because a fraction is divided by multiplying its denominator; and three-fifths of $\frac{4}{7}$ = 3 times $\frac{4}{35} = \frac{12}{35}$.

Rule.—*Multiply all the numerators together for a new numerator, and all the denominators together for a new denominator.*

NOTE 1.—If there are whole or mixed numbers in the compound expression, first reduce them to the form of a fraction.

NOTE 2.—If the result is desired in its lowest terms, cancel first all the factors common to the numerators and denominators.

NOTE 3.—This is reducing a compound fraction to a simple fraction.

EXAMPLES FOR PRACTICE.

Find the value

2. Of $\frac{4}{5}$ of $\frac{5}{8}$. Ans. $\frac{1}{2}$.
3. Of $\frac{3}{8}$ of $\frac{12}{15}$.
4. Of $\frac{8}{11}$ of $\frac{33}{40}$.
5. Of $\frac{5}{6}$ of $19\frac{1}{5}$.
6. Of $\frac{3}{10}$ of $3\frac{1}{3}$.
7. Of $\frac{3}{4}$ of $\frac{4}{5}$ of $\frac{5}{12}$.
8. Of $\frac{9}{16}$ of $\frac{49}{51}$ of $\frac{48}{63}$.
9. Of $\frac{3}{5}$ of $\frac{5}{6}$ of $\frac{8}{9}$ of $\frac{9}{12}$ of $\frac{18}{24}$. Ans. $\frac{4}{9}$.
10. Of $\frac{7}{8}$ of $3\frac{3}{4}$ of $1\frac{3}{5}$ of $1\frac{8}{5}$ of $6\frac{2}{3}$.
11. Of $\frac{2}{5}$ of $2\frac{1}{7}$ of $5\frac{1}{4}$ of $\frac{5}{8}$ of $\frac{28}{45}$. Ans. $1\frac{3}{4}$.

CASE IX.

Art. 137. To reduce fractions having different denominators to fractions having a common denominator.

NOTE.—Fractions having the same denominator are said to have a *common denominator*. Thus, $\frac{4}{7}$ and $\frac{5}{7}$ have a common denominator 7.

Ex. 1. Reduce $\frac{2}{3}$ and $\frac{5}{6}$ to fractions having like denominators.

SUGGESTION.—By inspection we see that $\frac{2}{3}$ becomes sixths by multiplying both terms by 2. Hence the answer may be $\frac{4}{6}$ and $\frac{5}{6}$.

Ex. 2. Reduce $\frac{5}{7}$ and $\frac{8}{9}$ to a common denominator.

One answer, $\frac{45}{63}$, $\frac{56}{63}$.

EXPLANATION.—Since *sevenths* cannot be reduced to *ninths*, nor *ninths* to *sevenths*, both fractions must be reduced to some denominator which is divisible by both 7 and 9. We may select 7 times $9 = 63$ for such denominator. Now, to raise $\frac{5}{7}$ to 63ds, both terms must be multiplied by 9, making $\frac{45}{63}$. (See Art. 129.) Also, to raise $\frac{8}{9}$ to 63ds, both terms must be multiplied by 7, making $\frac{56}{63}$.

Rule.—*Multiply the terms of each fraction by the denominators of all the other fractions.*

NOTE.—Afterward, if desired, reduce to a proposed common denominator by Article 129 or 130.

Examples for Practice.

Reduce to a common denominator.

3. $\frac{5}{6}$ and $\frac{8}{12}$. Ans. By inspection, $\frac{5}{6}$, $\frac{4}{6}$.
4. $\frac{4}{5}$ and $\frac{5}{6}$. Ans. $\frac{24}{30}$, $\frac{25}{30}$.
5. $\frac{6}{7}$ and $\frac{5}{8}$.
6. $\frac{5}{9}$, $\frac{7}{10}$, and $\frac{2}{15}$.
7. $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{1}{5}$.
8. $\frac{5}{6}$, $\frac{3}{5}$, and $\frac{1}{2}$.
9. $\frac{3}{4}$, $\frac{5}{7}$, and $\frac{3}{8}$.
10. $\frac{5}{8}$, $\frac{1}{2}$, $\frac{2}{5}$, $1\frac{2}{3}$, $12\frac{1}{2}$, and $7\frac{1}{7}$.

Ans. $\frac{2100}{3360}$, $\frac{1680}{3360}$, $\frac{1344}{3360}$, $\frac{5600}{3360}$, $\frac{42000}{3360}$, $\frac{24000}{3360}$.

11. $\frac{1}{11}$, $\frac{2}{13}$, $\frac{2}{5}$, $1\frac{1}{2}$, and $16\frac{2}{3}$.
12. $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{11}$ and $\frac{1}{10}$.
13. $\frac{1}{2}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{1}{20}$, $14\frac{2}{7}$, and 3.

Case X.

Art. 138. To reduce two or more fractions to their least common denominator.

Note.—The least common denominator of two or more fractions is the least integer that can be a denominator of all of them. When the fractions are reduced to it, the numerators are prime to each other.

Ex. 1. Reduce $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{7}{8}$ to their least common denominator.

Suggestion.—By inspection we see that 8 is the least number that can contain all of the denominators, and that, since one denominator is already 8, we need only raise the other fractions to eighths by Art. 129. The fractions then become $\frac{4}{8}$, $\frac{6}{8}$ and $\frac{7}{8}$.

Ex. 2. Reduce $\frac{7}{10}$, $\frac{11}{12}$, and $\frac{8}{15}$ to their least common denominator.

Explanation.—The least common denominator must be the least common multiple of the denominators 10, 12, and 15, which is 60. Now, reducing the fractions to 60ths by Art. 146, they become $\frac{42}{60}$, $\frac{55}{60}$, and $\frac{32}{60}$.

Written Process.

$$\begin{array}{r|rrr} 5 & 10 & 12 & 15 \\ \hline 3 & 2 & 12 & 3 \\ \hline 2 & 2 & 4 & 1 \\ \hline & 1 & 2 & 1 \end{array}$$

$5 \times 3 \times 2 \times 2 = 60.$

$60 \div 10 = 6 : 6 \times 7 = 42$, 1st Num.
$60 \div 12 = 5 : 5 \times 11 = 55$, 2d "
$60 \div 15 = 4 : 4 \times 8 = 32$, 3d "

Ans. $\frac{42}{60}$, $\frac{55}{60}$, $\frac{32}{60}$.

Rule.—*If one denominator is a multiple of each of the others, reduce the other fractions to that denominator.*

If no denominator is a multiple of every other, find the least common multiple of all the denominators, and write it as a new denominator; then reduce the fractions to this denominator.

NOTE.—First reduce to their lowest terms such fractions as are not already in them; also compound fractions to simple, and mixed or whole numbers to fractions.

EXAMPLES FOR PRACTICE.

Reduce to their least common denominator.

3. $\frac{1}{2}$, $\frac{5}{6}$, and $\frac{8}{12}$. Ans. By inspection, $\frac{3}{6}$, $\frac{5}{6}$, $\frac{4}{6}$.
4. $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{5}{6}$.
5. $\frac{3}{5}$, $\frac{8}{9}$, and $\frac{7}{10}$.
6. $\frac{5}{14}$, $\frac{10}{24}$, $\frac{19}{21}$, and $\frac{23}{42}$.
7. $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$, $\frac{5}{12}$, and $\frac{19}{24}$.
8. $\frac{5}{9}$, $\frac{11}{18}$, $\frac{7}{24}$, $\frac{31}{50}$, and $6\frac{1}{4}$.
9. $\frac{5}{8}$, $\frac{2}{9}$, $2\frac{1}{12}$, and $\frac{6}{11}$ of $\frac{33}{72}$.
10. $2\frac{1}{4}$, $3\frac{1}{2}$, $4\frac{1}{3}$, $1\frac{1}{6}$ and $\frac{3}{7}$.
11. $5\frac{3}{8}$, $7\frac{1}{4}$, and $\frac{3}{4}$ of $\frac{6}{7}$.

CASE XI.

Art. 139. To reduce two or more fractions to a common numerator.

Ex. 1. Reduce $\frac{3}{4}$, $\frac{5}{6}$, and $\frac{6}{7}$ to the same numerator.

One answer, $\frac{90}{120}$, $\frac{90}{108}$, $\frac{90}{105}$.

WRITTEN PROCESS.

$$\frac{3 \times 5 \times 6}{4 \times 5 \times 6} = \frac{90}{120}$$

$$\frac{5 \times 3 \times 6}{6 \times 3 \times 6} = \frac{90}{108}$$

$$\frac{6 \times 3 \times 5}{7 \times 3 \times 5} = \frac{90}{105}$$

EXPLANATION.

One multiple of the numerators is the product of them, 90. Hence 90 can be put for the new numerators. Then the given fractions are raised to this new numerator by multiplying the terms of each by the other two numerators, or by the method of Art. 129.

Rule.—*Multiply the terms of each fraction by the numerator, of all the other fractions.*

NOTE.—Afterward, if desired, reduce to a proposed common numerator by Articles 129 or 130.

EXAMPLES FOR PRACTICE.

Reduce to a common numerator

2. $\frac{3}{4}$, $\frac{5}{6}$, and $\frac{4}{5}$. Ans. $\frac{60}{80}$, $\frac{60}{72}$, $\frac{60}{75}$.
3. $\frac{5}{8}$, $\frac{2}{3}$, and $\frac{3}{7}$.
4. $\frac{2}{5}$, $\frac{3}{7}$, $\frac{4}{9}$, and $\frac{5}{11}$. Ans. $\frac{120}{300}$, $\frac{120}{280}$, $\frac{120}{270}$, $\frac{120}{264}$.
5. $\frac{5}{9}$, $\frac{6}{7}$, $\frac{7}{8}$, $4\frac{2}{7}$, and $3\frac{3}{4}$.
6. $\frac{8}{9}$, $\frac{12}{13}$, $\frac{6}{11}$, and $1\frac{1}{3}$.
7. $\frac{14}{15}$, $\frac{7}{20}$, $\frac{28}{33}$, and $\frac{21}{50}$.
8. $3\frac{3}{4}$, $12\frac{1}{2}$, $56\frac{1}{4}$, and $6\frac{3}{7}$.
9. $2\frac{2}{5}$, $16\frac{2}{3}$, 12, 7, and $18\frac{3}{4}$.

CASE XII.

Art. 140. To reduce two or more fractions to their least common numerator.

NOTE.—The least common numerator of two or more fractions is the least integer that can be a numerator of all of them. When the fractions are reduced to it, the denominators are prime to each other.

Ex. 1. Reduce $\frac{3}{4}$, $\frac{5}{6}$, and $\frac{6}{7}$ to their least common numerator.

Ans. $\frac{30}{40}$, $\frac{30}{36}$, $\frac{30}{35}$.

WRITTEN PROCESS.

3) 3 5 6	$30 \div 3 = 10 : 10 \times 4 = 40$, 1st denom.
	$30 \div 5 = \ \ 6 : \ \ 6 \times 6 = 36$, 2d	"
1 5 2	$30 \div 6 = \ \ 5 : \ \ 5 \times 7 = 35$, 3d	"
$3 \times 5 \times 2 = 30$		

EXPLANATION.—The least common numerator must be the least common multiple of the numerators 3, 5, and 6, which is 30. Now, reducing the fractions to numerator 30, by Art, 129, they become $\frac{30}{40}$, $\frac{30}{36}$, and $\frac{30}{35}$.

Rule.—*If one numerator is a multiple of each of the others, reduce the other fractions to that numerator.*

If no numerator is a multiple of every other, find the least common multiple of all the numerators, and write it as a new numerator; then reduce the fractions to this numerator.

NOTE.—First reduce to their lowest terms such fractions as are not already in them; also compound fractions to simple, and mixed or whole numbers to fractions.

EXAMPLES FOR PRACTICE.

Reduce to their least common numerator

2. $\frac{1}{3}$, $\frac{4}{12}$, and $\frac{5}{25}$. Ans. By inspection, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$.
3. $\frac{3}{4}$, $\frac{6}{7}$, and $\frac{12}{13}$. $\frac{12}{16}$, $\frac{12}{14}$, $\frac{12}{13}$.
4. $\frac{2}{3}$, $\frac{3}{5}$, $\frac{4}{7}$, $\frac{5}{6}$, and $\frac{2}{9}$.
5. $\frac{5}{8}$, $\frac{10}{21}$, and $\frac{15}{16}$.
6. $\frac{8}{9}$, $\frac{6}{7}$, $\frac{12}{15}$, and $\frac{36}{45}$.
7. $\frac{6}{7}$, $\frac{5}{8}$, $\frac{10}{17}$, and $\frac{15}{23}$.
8. $\frac{5}{7}$, $\frac{8}{9}$, $\frac{10}{11}$, and $6\frac{2}{3}$.
9. $1\frac{1}{4}$, $2\frac{2}{3}$, $8\frac{4}{7}$, and $2\frac{1}{7}$.
10. $12\frac{1}{2}$, $16\frac{2}{3}$, $18\frac{3}{4}$, and 30.

NOTE.—For the reduction of a complex to a simple fraction, which would form Case XIII of the Reduction of Fractions, see Art. 158.

ADDITION OF COMMON FRACTIONS.

Art. 141. Addition of fractions is the process of uniting two or more fractions so as to form one fraction which is equivalent to them, called their *sum* or *amount.*

In whole numbers, only units of the same kind are capable of being added. (See Art. 32.) In fractions, only like parts of like units are capable of being added. Thus 3 *eighths* and 2 *sevenths* are neither 5 *eighths*, nor 5 *sevenths*, nor 5 *fifteenths*. But, if $\frac{3}{8}$ and $\frac{2}{7}$ are reduced to fifty-sixths, making $\frac{21}{56}$ and $\frac{16}{56}$, it is plain that their sum is 37 fifty-sixths, or $\frac{37}{56}$. Also, it is plain that these must be fractions either of the same abstract unit or of the same denominate unit. Thus, $\frac{21}{56}$ of a day cannot be united with $\frac{16}{56}$ of a mile, because one of these fractions is a fraction of a unit of time, and the other is a fraction of a unit of distance.

Art. 142. To find the sum of two or more fractions.

Ex. 1. What is the sum of $\frac{3}{7}$ and $\frac{2}{7}$? Ans. $\frac{5}{7}$.

PROCESS INDICATED.

$$\frac{3}{7}+\frac{2}{7}=\frac{3+2}{7}=\frac{5}{7}.$$

EXPLANATION.

Since 3 and 2 are 5, three sevenths and two sevenths must be five sevenths.

Ex. 2. What is the sum of $\frac{1}{3}$ and $\frac{3}{5}$? Ans. $\frac{14}{15}$.

PROCESS INDICATED.

$$\frac{1}{3}+\frac{3}{5}=\frac{5}{15}+\frac{9}{15}=\frac{5+9}{15}=\frac{14}{15}.$$

EXPLANATION.

Since *thirds* and *fifths* are not like parts of a unit, they cannot be united in one sum. Reducing both fractions to fifteenths, $\frac{1}{3}$ is $\frac{5}{15}$, and $\frac{3}{5}$ is $\frac{9}{15}$, and their sum is $\frac{14}{15}$.

Ex. 3. Find the sum of $5\frac{5}{12}$, $3\frac{5}{6}$, and $10\frac{5}{8}$. Ans. $19\frac{7}{8}$.

FIRST METHOD.

$5\frac{5}{12} = \frac{65}{12}, = \frac{130}{24}.$

$3\frac{5}{6} = \frac{23}{6}, = \frac{92}{24}.$

$10\frac{5}{8} = \frac{85}{8}, = \frac{255}{24}.$

Sum, $\frac{477}{24} = 19\frac{21}{24}, = 19\frac{7}{8}.$

SECOND METHOD.

$5\frac{5}{12} = 5\frac{10}{24}.$

$3\frac{5}{6} = 3\frac{20}{24}.$

$10\frac{5}{8} = 10\frac{15}{24}.$

Sum, $18\frac{45}{24}, = 19\frac{21}{24}, = 19\frac{7}{8}.$

EXPLANATION.

In the first method, the mixed numbers are reduced to fractions; then these fractions are reduced to a common denominator, and added.

In the second method, the whole numbers are added by themselves, the sum being 18: the fractions are added by themselves, the sum being $\frac{45}{24}, = 1\frac{21}{24}$: then the sum of 18 and $1\frac{21}{24}$ is $19\frac{21}{24} = 19\frac{7}{8}$.

Rule.—*If the fractions have a common denominator, add their numerators, and write the sum over the denominator.*

If the fractions have different denominators, reduce them first to a common denominator, and write the sum of the new numerators over the common denominator.

Either reduce mixed numbers to fractions, and add them as fractions, or find the sums of the integral and fractional portions separately, and add these sums.

NOTE 1.—To lessen labor, first reduce to their lowest terms such fractions as are not already in them; also, compound to simple fractions.

NOTE 2.—In adding fractions whose numerators are a unit, the student will perceive that *their sum is equal to the sum of the denominators over their product.* (See Examples 20 and 30.) When the numerators are alike, but not a unit, *the sum of the fractions is equal to the product of the sum of their denominators by their numerator, over the product of the denominators.* (See Example 21.)

EXAMPLES FOR PRACTICE.

What is the sum

4. Of $\frac{3}{4}$, and $\frac{2}{4}$? Ans. $1\frac{1}{4}$.
5. Of $\frac{5}{6}$, $\frac{7}{6}$, and $\frac{9}{6}$?
6. Of $\frac{3}{7}$, $\frac{4}{7}$, $\frac{6}{7}$, and $\frac{1}{7}$?
7. Of $\frac{5}{12}$, $\frac{6}{12}$, $\frac{7}{12}$, and $\frac{11}{12}$?
8. Of $\frac{3}{16}$, $\frac{4}{16}$, $\frac{11}{16}$, $\frac{12}{16}$, and $\frac{15}{16}$? Ans. $2\frac{13}{16}$.
9. Of $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{5}{6}$? Ans. $2\frac{1}{4}$.
10. Of $\frac{1}{3}$, $\frac{3}{4}$, and $\frac{7}{8}$?
11. Of $\frac{5}{6}$, $\frac{7}{8}$, and $\frac{11}{12}$?
12. Of $\frac{2}{3}$, $\frac{5}{8}$, and $\frac{7}{10}$?
13. Of $\frac{3}{10}$, $\frac{2}{3}$, and $\frac{14}{15}$?
14. Of $\frac{1}{2}$, and $\frac{3}{4}$ of $\frac{5}{6}$? Ans. $1\frac{1}{8}$
15. Of $\frac{2}{3}$, and $\frac{2}{5}$ of $\frac{5}{8}$? Ans. $\frac{11}{12}$.
16. Of $\frac{2}{3}$ of $\frac{9}{10}$, and $\frac{9}{10}$? Ans. $1\frac{1}{2}$.
17. Of $\frac{7}{8}$, and $\frac{2}{3}$ of $\frac{15}{16}$?
18. Of $\frac{11}{12}$, and $\frac{5}{6}$ of $\frac{9}{10}$?
19. Of $\frac{5}{8}$, $\frac{9}{10}$, $\frac{5}{6}$, and $\frac{2}{5}$?
20. Of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$?
21. Of $\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{7}$, and $\frac{2}{10}$?
22. Of $\frac{5}{8}$, and $\frac{3}{4}$ of $7\frac{1}{2}$?
23. Of $\frac{3}{7}$, and $\frac{2}{3}$ of $3\frac{1}{3}$?
24. Of $6\frac{3}{4}$, $7\frac{5}{8}$, $16\frac{5}{16}$, and 10? Ans. $40\frac{11}{16}$.
25. Of $2\frac{1}{2}$, $3\frac{1}{3}$, $4\frac{5}{6}$, and $10\frac{1}{12}$?
26. Add $\frac{4}{5}$ of $\frac{5}{7}$, $\frac{1}{2}$ of $\frac{2}{3}$, and $\frac{5}{6}$.
27. Add $\frac{3}{5}$, $\frac{4}{5}$ of $\frac{5}{8}$, $\frac{11}{31}$, of $\frac{62}{99}$, and $\frac{7}{12}$.
28. $1\frac{1}{3} + 1\frac{1}{4} + 1\frac{1}{5} + 1\frac{5}{6} =$?
29. $3\frac{1}{3} + 6\frac{1}{4} + 12\frac{1}{2} + 18\frac{3}{4} + 31\frac{1}{4} + 37\frac{1}{2} =$?
30. A boy paid $\frac{1}{8}$ of a dollar for a knife, $\frac{1}{6}$ of a dollar for a slate, and $\frac{1}{4}$ of a dollar for an arithmetic; how much did they all cost?
31. What number is that from which if $5\frac{2}{3}$ be taken, the remainder will be $10\frac{2}{5}$? Ans. $16\frac{1}{15}$.
32. Four farms contain respectively $92\frac{7}{8}$, $124\frac{2}{3}$, $140\frac{11}{16}$, and $56\frac{1}{4}$ acres; what do they all contain? Ans. $414\frac{23}{48}$ acres.

SUBTRACTION OF COMMON FRACTIONS.

Art. 143. Subtraction of fractions is the process of finding the difference between two fractions, or between a fraction and a whole number.

In whole numbers, the minuend, subtrahend, and difference must be expressed in like units. (See Art. 37.) In fractions, the minuend, subtrahend, and difference must be expressed in like parts of like units. Thus, *one-half* cannot be directly taken from *three-fourths;* but, if one-half is reduced to fourths, making two-fourths, it can be taken from three-fourths, leaving one-fourth. Also, two-fourths of a pound cannot be taken from three-fourths of an hour, because the former is a fraction of a unit of weight, and the latter, of time.

Art. 144. To find the difference between two fractions.

Ex. 1. From $\frac{7}{9}$ take $\frac{5}{9}$. Ans. $\frac{2}{9}$.

PROCESS INDICATED.

$$\frac{7}{9}-\frac{5}{9}=\frac{7-5}{9}=\frac{2}{9}.$$

EXPLANATION.

Since 5 units from 7 units, leaves 2 units, 5 ninths from 7 ninths must leave 2 ninths.

Ex. 2. From $\frac{7}{9}$ take $\frac{1}{2}$. Ans. $\frac{5}{18}$.

PROCESS INDICATED.

$$\frac{7}{9}-\frac{1}{2}=\frac{14}{18}-\frac{9}{18}=\frac{14-9}{18}=\frac{5}{18}.$$

EXPLANATION.

Since *ninths* and halves are not like parts of a unit, $\frac{1}{2}$ cannot be directly taken from $\frac{7}{9}$; but, reducing both to *eighteenths,* they become $\frac{14}{18}$ and $\frac{9}{18}$, and their difference is $\frac{5}{18}$.

Ex. 3. From $4\frac{5}{6}$ take $2\frac{3}{4}$. Ans. $2\frac{1}{12}$.

REMARK.—The following methods are sufficiently clear without any extended explanation.

FIRST METHOD.	SECOND METHOD.
$4\frac{5}{6} = \frac{29}{6} = \frac{58}{12}$	$4\frac{5}{6} = 4\frac{10}{12}$
$2\frac{3}{4} = \frac{11}{4} = \frac{33}{12}$	$2\frac{3}{4} = 2\frac{9}{12}$
$\frac{25}{12} = 2\frac{1}{12}$.	$2\frac{1}{12}$.

Ex. 4. From 9 take $\frac{2}{5}$. Ans. $8\frac{3}{5}$.

FIRST METHOD.	SECOND METHOD.
$9 = \frac{45}{5}$	9
$\frac{2}{5} = \frac{2}{5}$	$\frac{2}{5}$
$\frac{43}{5} = 8\frac{3}{5}$.	$8\frac{3}{5}$.

EXPLANATION.

In the second method we can proceed thus:—$\frac{2}{5}$ from 0, impossible; change 9 into its equivalent $8\frac{5}{5}$; then $\frac{2}{5}$ from $\frac{5}{5}$ leaves $\frac{3}{5}$, and 0 from 8 leaves 8. Or we can proceed thus:—$\frac{2}{5}$ from 0, impossible; add $\frac{5}{5}$ to 9, making $9\frac{5}{5}$; then $\frac{2}{5}$ from $\frac{5}{5}$ leaves $\frac{3}{5}$; write $\frac{3}{5}$, and carry 1 to 0, making 1; 1 from 9 leaves 8, making the whole remainder $8\frac{3}{5}$.

Ex. 5. From $6\frac{1}{3}$ take $4\frac{3}{4}$. Ans. $1\frac{7}{12}$.

FIRST METHOD.	SECOND METHOD.
$6\frac{1}{3} = \frac{19}{3} = \frac{76}{12}$	$6\frac{1}{3} = 6\frac{4}{12}$
$4\frac{3}{4} = \frac{19}{4} = \frac{57}{12}$	$4\frac{3}{4} = 4\frac{9}{12}$
Remainder $\frac{19}{12}$, $= 1\frac{7}{12}$.	$1\frac{7}{12}$.

EXPLANATION.

In the second method we can proceed thus:—$\frac{9}{12}$ from $\frac{4}{12}$ impossible; change $6\frac{4}{12}$ to its equivalent $5\frac{16}{12}$; then $\frac{9}{12}$ from $\frac{16}{12}$ leaves $\frac{7}{12}$, and 4 from 5 leaves 1.

Or we can proceed thus;—$\frac{9}{12}$ from $\frac{4}{12}$ impossible; add $\frac{12}{12}$ to $\frac{4}{12}$, making $\frac{16}{12}$; then $\frac{9}{12}$ from $\frac{16}{12}$ leaves $\frac{7}{12}$; write the $\frac{7}{12}$, and carry 1 to 4, making 5 : 5 from 6 leaves 1, making the whole remainder $11\frac{7}{12}$.

Ex. 6. From $7\frac{1}{5}$ take $\frac{2}{3}$. Ans. $6\frac{8}{15}$.

FIRST METHOD.	SECOND METHOD.
$7\frac{1}{5} = \frac{36}{5} = \frac{108}{15}$	$7\frac{1}{5} = 7\frac{3}{15}$
$\frac{2}{3} = \frac{10}{15}$	$\frac{2}{3} = 0\frac{10}{15}$
Remainder, $\frac{98}{15} = 6\frac{8}{15}$.	$6\frac{8}{15}$.

EXPLANATION.

In the second method we can proceed as in the second method of Ex. 5, carrying 1 to 0 in the last style of managing the subtraction, or not writing the 0, and subtracting the carried 1 from the upper figure.

Rule.—*If the fractions have a common denominator, subtract the less numerator from the greater, and write the difference over the denominator.*

If the fractions have different denominators, reduce them to a common denominator, then subtract as before stated.

Either reduce mixed numbers to fractions, and subtract as directed for fractions; or

Write the subtrahend under the minuend, and reduce the fractional parts to a common denominator. If the upper fraction is greater than the lower, subtract, and write the difference below; then subtract the lower whole number from the upper. But if the upper number has either no fraction, or one smaller than the lower fraction, add to the upper place or fraction as many parts, like those of the lower fraction, as make a unit; then subtract the lower fraction, write the difference, and carry 1 to the lower whole number.

NOTE 1.—To lessen labor, first reduce to their lowest terms those fractions which are not already in them, and compound to simple fractions.

NOTE 2.—In subtracting one fraction from another when their *numerators* are a unit, the student will perceive that *their difference is equal to the difference of the denominators over their product.* (See Examples 18 and 19.) When the numerators are alike, but not a unit, the difference of the fractions is equal to *the product of the difference of the denominators by their numerator, over the product of the denominators.* (See Examples 20 and 21.)

EXAMPLES FOR PRACTICE.

From

7. $\frac{7}{8}$ take $\frac{5}{8}$. Ans. $\frac{1}{4}$.
8. $\frac{9}{11}$ take $\frac{3}{11}$. Ans. $\frac{6}{11}$.
9. $\frac{11}{12}$ take $\frac{5}{12}$.
10. $\frac{6}{7}$ take $\frac{4}{7}$.
11. $\frac{4}{5}$ take $\frac{2}{3}$. Ans. $\frac{2}{15}$.
12. $\frac{5}{8}$ take $\frac{2}{5}$.
13. $\frac{11}{15}$ take $\frac{7}{24}$.
14. $\frac{17}{26}$ take $\frac{19}{65}$.
15. $\frac{23}{36}$ take $\frac{7}{24}$.
16. $12\frac{1}{2}$ take $4\frac{11}{18}$. Ans. $7\frac{8}{9}$.
17. $16\frac{2}{3}$ take $7\frac{3}{5}$.

From

18. $\frac{1}{2}$ take $\frac{1}{5}$. Ans. $\frac{3}{10}$.
19. $\frac{1}{3}$ take $\frac{1}{7}$.
20. $\frac{2}{5}$ take $\frac{2}{7}$. Ans. $\frac{4}{35}$.
21. $\frac{3}{4}$ take $\frac{3}{5}$.
22. $7\frac{2}{3}$ take $2\frac{5}{8}$.
23. $10\frac{2}{7}$ take $5\frac{5}{6}$.
24. $\frac{3}{4}$ of $\frac{8}{9}$ take $\frac{2}{7}$. Ans. $\frac{8}{21}$.
25. $\frac{3}{5}$ of $6\frac{1}{4}$ take $1\frac{3}{8}$.
26. $12\frac{2}{5}$ take $\frac{5}{6}$ of $10\frac{1}{6}$.
27. 12 take $\frac{1}{2}$ of $\frac{3}{7}$.
28. 15 take $\frac{2}{5}$ of $8\frac{1}{3}$.

29. The sum of two numbers is $21\frac{3}{4}$, and the less is $8\frac{3}{7}$; *what is the greater?*

30. A man having $\$37\frac{1}{2}$, spent $\$20\frac{3}{7}$; how much had he left?

31. From a piece of ribbon containing $23\frac{3}{5}$ yards, $16\frac{2}{3}$ yards were sold; how much remained?

32. To what fraction must $\frac{2}{9}$ be added, that the sum may be $\frac{7}{8}$?

33. To what number must $\frac{7}{8}$ be added, that the sum may be $3\frac{1}{8}$?

MULTIPLICATION OF COMMON FRACTIONS.

Art. 145. Multiplication of fractions is the process of finding the product of a whole and fractional number, or of two fractional numbers.

To multiply by a whole number is to find the sum of as many expressions of the multiplicand as there are units in the multiplier. Thus, to multiply 5 by 2 is to find the sum of two expressions of 5, which is 10.

To multiply by a fraction is to find that part of the multiplicand which the multiplier is of a unit. Thus, to multiply 8 by $\frac{3}{4}$ is to find that part of 8 which $\frac{3}{4}$ is of 1, which is $\frac{3}{4}$ of 8, that is, 3 times $\frac{1}{4}$ of 8, or 3 times 2, which is 6.

Since the product of two factors is the same, whichever is the multiplier, *the product of a whole number by a fraction is the same as the product of that fraction by the whole number.* Thus, $\frac{3}{4}$ of 8 is equal to 8 times $\frac{3}{4}$.

CASE I.

Art. 146. To multiply a fraction by a whole number.

Ex. 1. If a pound of sugar costs $\frac{1}{10}$ of a dollar, how much does 5 pounds cost? Ans, $\frac{1}{2}$ of a dollar.

FIRST METHOD.

$$\tfrac{1}{10} \times 5 = \frac{1 \times 5}{10} = \frac{5}{10}, = \frac{1}{2}.$$

EXPLANATION.

If 1 pound of sugar costs $\frac{1}{10}$ of a dollar, 5 pounds cost 5 times 1 tenth of a dollar, that is $\frac{5}{10}$ of a dollar, equal to $\frac{1}{2}$ of a dollar.

Second Method.

$$\tfrac{1}{10} \times 5 = \frac{1}{10 \div 5} = \frac{1}{2}.$$

Explanation.

If 1 pound costs $\frac{1}{10}$ of a dollar, 5 pounds cost one of those parts of a dollar which are 5 times as large as *tenths*, that is *one-half* of a dollar. Such a part is found by dividing the denominator of $\frac{1}{10}$ by 5, according to Art. 123.

Rule.—*Multiply the numerator, and write the product over the denominator;* or

Divide the denominator by the multiplier, when it can be done without a remainder, and write the quotient as a new denominator to the given numerator.

Examples for Practice.

Multiply		Multiply.
2. $\frac{3}{8}$ by 2.	Ans. $\frac{3}{4}$.	10. $\frac{9}{21}$ by 7.
3. $\frac{7}{8}$ by 4.	Ans. $3\frac{1}{2}$.	11. $\frac{41}{75}$ by 25.
4. $\frac{11}{15}$ by 3.		12. $\frac{23}{25}$ by 10.
5. $\frac{13}{45}$ by 5.		13. $\frac{12}{19}$ by 7.
6. $\frac{3}{4}$ by 7.	Ans. $5\frac{1}{4}$.	14. $\frac{7}{11}$ by 12.
7. $\frac{3}{7}$ by 8.		15. $\frac{24}{35}$ by 16.
8. $\frac{5}{11}$ by 12.		16. $\frac{33}{40}$ by 35.
9. $\frac{16}{25}$ by 10.	Ans. $6\frac{2}{3}$.	17. $\frac{5}{8}$ by 20.

18. What cost 4 pounds butter at $\$\frac{5}{8}$ a pound? Ans. $\$2\frac{1}{2}$.
19. What cost 11 pounds coffee at $\$\frac{2}{7}$ a pound?
20. At $\$\frac{5}{8}$ per bushel what cost 60 bushel of oats?
21. At $\$\frac{13}{60}$ per dozen what cost 12 dozen of eggs?
22. What cost 6 caps at $\$\frac{5}{8}$ apiece?
23. What cost 63 bushels of potatoes at $\$\frac{2}{5}$ per bushel?
24. At $\$\frac{11}{50}$ per pound what cost 200 pounds of fish?

Case II.

Art. 147. To multiply a whole number by a fraction.

Ex. 1. If 1 yard of flannel costs 60 cents, how much will $\frac{3}{4}$ of a yard cost? Ans. 45 cents.

FIRST METHOD.	SECOND METHOD.
$\begin{array}{r} 60 \\ 3 \\ \hline 4\,)\,180 \\ \hline 45 \end{array}$	$\begin{array}{r} 4\,)\,60 \\ \hline 15 \\ 3 \\ \hline 45 \end{array}$

EXPLANATION.

If 1 yard of flannel cost 60 cents, $\frac{3}{4}$ of a yard costs $\frac{3}{4}$ of 60 cents, which may be found in two ways:—

FIRST.—$\frac{3}{4}$ of 60 cents equals $\frac{1}{4}$ of 3 times 60 cents, that is, $\frac{1}{4}$ of 180 cents, or 45 cents.

SECOND.—$\frac{3}{4}$ of 60 cents equals 3 times $\frac{1}{4}$ of 60 cents, that is, 3 times 15 cents, or 45 cents.

NOTE.—The second method is most quickly apprehended as true, and is preferable when the division is exact.

Rule.—*Either multiply the whole number by the numerator, and divide the product by the denominator;* or

Divide the whole number by the denominator, and multiply the quotient by the numerator.

EXAMPLES FOR PRACTICE.

Multiply

2. 12 by $\frac{3}{4}$. Ans. 9.
3. 20 by $\frac{2}{5}$.
4. 24 by $\frac{5}{6}$.
5. 33 by $\frac{7}{11}$.
6. 102 by $\frac{6}{17}$. Ans. 36.
7. 144 by $\frac{5}{12}$.
8. 200 by $\frac{2}{5}$.
9. 285 by $\frac{10}{19}$. Ans. 150.

Multiply

10. 7 by $\frac{3}{4}$. Ans. $5\frac{1}{4}$.
11. 8 by $\frac{3}{5}$. Ans. $4\frac{4}{5}$.
12. 11 by $\frac{2}{3}$.
13. 20 by $\frac{3}{7}$.
14. 31 by $\frac{13}{25}$.
15. 65 by $\frac{11}{21}$.
16. 83 by $\frac{3}{16}$. Ans. $15\frac{9}{16}$.
17. 105 by $\frac{12}{45}$.

18. What cost $\frac{3}{4}$ of ton hay at \$24 per ton? Ans. \$18.
19. What cost $\frac{7}{8}$ of an acre land at \$130 per acre?
20. What cost $\frac{3}{5}$ of a barrel flour at \$10 per barrel?
21. What cost $\frac{3}{7}$ of a yard cloth at \$6 per yard?
22. What cost $\frac{5}{8}$ of a barrel fish at \$23 per barrel?
23. What cost $\frac{2}{3}$ of a pound coffee at 27 cents per pound?
24. What cost $\frac{3}{4}$ of a ton coal at \$5 per ton?
25. What cost $\frac{5}{9}$ of a barrel apples at \$4 per barrel?
26. What cost the building of $\frac{7}{10}$ of a mile of railroad at the rate of \$15825 per mile?

CASE III.

Art. 148. To multiply a mixed number by a whole number.

Since the whole product is the sum of the products of all parts of the multiplicand by the multiplier, we have the following

Rule.—*Multiply separately the integral and fractional parts of the multiplicand, and add the products;* or

Reduce the multiplicand to an improper fraction, and multiply it by the multiplier.

EXAMPLES FOR PRACTICE.

Multiply

1. $8\frac{2}{3}$ by 6. Ans. 52.
2. $4\frac{3}{5}$ by 7. Ans. $32\frac{1}{5}$.
3. $7\frac{3}{8}$ by 9.
4. $15\frac{3}{4}$ by 8.
5. $21\ \frac{3}{7}$ by 11.

Multiply

6. $17\frac{1}{2}$ by 25.
7. $43\frac{2}{4}$ by 123.
8. $65\frac{3}{4}$ by 30.
9. $31\frac{1}{4}$ by 13.
10. $24\frac{3}{4}$ by 15.

11. What cost 5 barrels flour at $\$11\frac{3}{4}$ per barrel? Ans. $\$58\frac{3}{4}$.

12. What cost 10 yards cloth at $\$5\frac{2}{7}$ per yard?

13. What cost 17 sheep $\$6\frac{3}{5}$ per head?

14. If a man walk $28\frac{5}{6}$ miles each day, how far will he walk in 50 days? Ans. $1441\frac{2}{3}$ miles.

15. What cost 16 barrels oil at $\$7\frac{9}{16}$ per barrel?

CASE IV.

Art. 149. To multiply an integer by a mixed number.

Since the whole product is equal to the sum of the products by all parts of the multiplier, we have the following

Rule.—*Multiply separately by the integral and fractional parts of the multiplier, and add the products;* or

Reduce the mixed number to an improper fraction, and then multiply by the fraction.

EXAMPLES FOR PRACTICE.

Multiply		Multiply
1. 24 by $8\frac{1}{3}$.	Ans. 200.	6. 25 by $20\frac{2}{3}$.
2. 35 by $7\frac{3}{5}$.	Ans. 266.	7. 83 by $7\frac{3}{10}$.
3. 41 by $8\frac{5}{6}$.		8. 91 by $12\frac{5}{13}$.
4. 63 by $10\frac{3}{4}$.		9. 102 by $6\frac{5}{11}$.
5. 72 by $9\frac{5}{7}$.		10. 423 by $132\frac{5}{7}$.

11. What cost $7\frac{3}{4}$ yards velvet at $5 per yard?

12. What cost $8\frac{3}{7}$ tons hay at $20 per ton?

13. There are 5280 feet in one mile, how many feet are there in $21\frac{3}{4}$ miles?

14. At $40 per acre what cost $157\frac{5}{8}$ acres land?

15. If the building of a railroad costs $15650 per mile; what is the cost of building $7\frac{3}{10}$ miles.

16. What cost $7\frac{3}{4}$ thousand feet of lumber at $36 per thousand? Ans. $279.

17. If a man walks 27 miles per day; how far can he walk in $13\frac{7}{10}$ days?

CASE V.

Art. 150. To multiply a fraction by a fraction.

Ex. 1. A man who owned $\frac{3}{5}$ of the stock of a mill sold $\frac{1}{3}$ of his portion. What part of the whole stock did he sell?

Ans. $\frac{2}{5}$.

PROCESS INDICATED.

$$\frac{2}{3} \times \frac{3}{5} = \frac{2 \times 3}{3 \times 5}, = \frac{6}{15} = \frac{2}{5}.$$

By Cancellation.

$$\frac{2 \times \cancel{3}}{\cancel{3} \times 5} = \frac{2}{5}$$

EXPLANATION.

He sold $\frac{2}{3}$ of $\frac{3}{5}$ of the whole stock. One-third of $\frac{3}{5}$ is $\frac{3}{5}$ divided by 3, which is done either by dividing the numerator, making $\frac{1}{5}$, or by multiplying the denominator, making $\frac{3}{15}$. Two-thirds of $\frac{3}{5}$ must be either 2 times $\frac{1}{5}$, making $\frac{2}{5}$, or 2 times $\frac{3}{15}$, making $\frac{6}{15}$, which is, in its lowest terms, $\frac{2}{5}$.

Rule.—*Multiply the numerators together for a new numerator and the denominators together for a new denominator.*

NOTE.—Abbreviate by canceling, when possible.

EXAMPLES FOR PRACTICE.

Multiply		Multiply	
2. $\frac{5}{6}$ by $\frac{3}{5}$.	Ans. $\frac{1}{2}$.	7. $\frac{6}{7}$ by $\frac{5}{8}$.	
3. $\frac{3}{5}$ by $\frac{6}{7}$.	Ans. $\frac{18}{35}$.	8. $\frac{5}{9}$ by $\frac{4}{13}$.	
4. $\frac{3}{7}$ by $\frac{14}{15}$.		9. $\frac{7}{20}$ by $\frac{5}{14}$ of $\frac{4}{11}$.	
5. $\frac{4}{5}$ by $\frac{25}{28}$.	Ans. $\frac{5}{7}$.	10. $\frac{16}{25}$ by $\frac{5}{8}$ of $\frac{2}{3}$.	
6. $\frac{2}{3}$ by $\frac{11}{16}$.		11. $\frac{4}{5}$ by $\frac{5}{4}$.	Ans. 1.

12. What cost $\frac{3}{4}$ of a yard lace at \$$\frac{3}{5}$ per yard? Ans. \$$\frac{9}{20}$.
13. What is the square of $\frac{5}{7}$?
14. What is the cube of $\frac{3}{5}$?
15. Multiply together $\frac{2}{3}$, $\frac{3}{4}$ of $\frac{8}{9}$, $\frac{21}{32}$, and $\frac{5}{7}$? Ans. $\frac{5}{24}$.

CASE VI.

Art. 151. To multiply one mixed number by another.

Ex. 1. Multiply $8\frac{2}{3}$ by $5\frac{3}{4}$. Ans. $49\frac{5}{6}$.

FIRST METHOD.

$$\left.\begin{array}{l} 8\frac{2}{3}=\frac{26}{3} \\ 5\frac{3}{4}=\frac{23}{4} \end{array}\right\} \frac{26}{3}\times\frac{23}{4}=\frac{598}{12}=49\frac{10}{12}=49\frac{5}{6}.$$

SECOND METHOD.

$$\begin{array}{lr} & 8\frac{2}{3} \\ & 5\frac{3}{4} \\ \hline 5\times 8= & 40 \\ 5\times\frac{2}{3}=\frac{10}{3}= & \frac{40}{12} \\ \frac{3}{4}\times 8=\frac{24}{4}= & \frac{72}{12} \\ \frac{3}{4}\times\frac{2}{3}= & \frac{6}{12} \\ \hline \end{array}$$

$$40\ \tfrac{118}{12}=40+9\tfrac{10}{12}=49\tfrac{10}{12}=49\tfrac{5}{6}.$$

EXPLANATION.

The first method consists in reducing the mixed numbers to improper fractions, and multiplying them together as fractions. The second method consists in multiplying every part of the multiplicand by every part of the multiplier, and adding the products.

Rule.—*Either reduce the mixed numbers to fractions, and multiply as in the case of a fraction by a fraction;* or

Multiply the whole number and fraction of the multiplicand separately, by the whole number and fraction of the multiplier separately, and unite the products in one sum.

EXAMPLES FOR PRACTICE.

Multiply	Multiply
2. $3\frac{1}{3}$ by $2\frac{1}{4}$. Ans. $7\frac{1}{2}$.	6. $22\frac{2}{5}$ by $3\frac{3}{5}$.
3. $6\frac{3}{5}$ by $1\frac{1}{4}$.	7. $33\frac{1}{3}$ by $\frac{3}{5}$ of $15\frac{2}{3}$.
4. $8\frac{3}{7}$ by $6\frac{2}{3}$.	8. $18\frac{3}{4}$ by $\frac{4}{5}$ of $20\frac{5}{8}$.
5. $12\frac{5}{6}$ by $7\frac{3}{5}$.	9. $20\frac{2}{3}$ by $30\frac{3}{4}$.

10. What cost $2\frac{1}{2}$ yards cloth at $\$6\frac{3}{5}$ per yard?
11. What cost $40\frac{1}{2}$ bushels wheat at $\$2\frac{3}{20}$ per bushel?
12. What is the square of $3\frac{1}{3}$?
13. What is the cube of $3\frac{3}{4}$?
14. Multiply the square of $1\frac{2}{3}$ by the cube of $1\frac{1}{4}$.
15. Multiply together $10\frac{1}{2}$, $20\frac{3}{4}$ and $30\frac{2}{3}$.
16. Find the value of $2\frac{2}{5} \times \frac{8}{11} \times 13\frac{1}{8} \times \frac{5}{108}\ 1\frac{7}{15} \times 1\frac{2}{7}$.
Ans. 2.

DIVISION OF COMMON FRACTIONS.

Art. 152. Division of fractions includes all cases of division in which a fractional number is divisor, or dividend, or both.

CASE I.

Art. 153. To divide a a fraction by a whole number.

Ex. 1. Divide $\frac{8}{9}$ by 2. Ans. $\frac{4}{9}$.

METHODS INDICATED.

FIRST.—$\frac{8}{9} \div 2 = \dfrac{8 \div 2}{9} = \frac{4}{9}$.

SECOND.—$\frac{8}{9} \div 2 = \dfrac{8}{2 \times 9}, = \frac{8}{18} = \frac{4}{9}$.

THIRD.—$\frac{8}{9} \div 2 = \frac{1}{2}$ of $\frac{8}{9} = \frac{8}{18} = \frac{4}{9}$.

EXPLANATION.

In the first method we divide $\frac{8}{9}$ by 2 by dividing the numerator, making $\frac{4}{9}$. (See Art. 122.)

In the second method we divide $\frac{8}{9}$ by 2 by multiplying the denominator by 2, making $\frac{8}{18}$, which, in lowest terms, is $\frac{4}{9}$.

In the third method we consider the division of $\frac{8}{9}$ by 2 as finding $\frac{1}{2}$ of $\frac{8}{9}$, which is finding the value of a compound fraction.

Rule.—*Divide the numerator by the whole number, and write the quotient over the denominator.* Or

Multiply the denominator by the whole number, and write the product under the numerator.

NOTE 1.—The former method is preferable when the numerator is exactly divisible by the whole number.

NOTE 2.—Abbreviate by canceling, when possible.

EXAMPLES FOR PRACTICE.

Divide		Divide	
2. $\frac{6}{7}$ by 2.	Ans. $\frac{3}{7}$.	8. $\frac{5}{8}$ by 3.	Ans. $\frac{5}{24}$.
3. $\frac{8}{11}$ by 4.		9. $\frac{13}{16}$ by 5.	
4. $\frac{10}{13}$ by 5.		10. $\frac{12}{25}$ by 11.	
5. $\frac{12}{25}$ by 4.		11. $\frac{16}{31}$ by 12.	
6. $\frac{121}{130}$ by 11.		12. $\frac{35}{57}$ by 20.	Ans. $\frac{7}{228}$.
7. $\frac{150}{171}$ by 30.		13. $\frac{81}{98}$ by 36.	Ans. $\frac{9}{392}$.

14. If 7 pounds of sugar cost $\$\frac{42}{50}$, how much is that per pound? Ans. $\$\frac{3}{25}$.

15. If 8 apples cost $\$\frac{5}{7}$, what will 1 cost? Ans. $\$\frac{5}{56}$.

CASE II.

Art. 154. To divide a mixed number by a whole number.

Ex. 1. Divide $12\frac{4}{5}$ by 4. Ans. $3\frac{1}{5}$.

FIRST METHOD.

$$4\ )\ 12\frac{4}{5}$$
$$\overline{\quad 3\frac{1}{5}.}$$

EXPLANATION.

In the first method we first divide the integral part, 12 by 4, making 3; then $\frac{4}{5}$ by 4 making $\frac{1}{5}$, and the whole quotient is $3\frac{1}{5}$.

SECOND METHOD.

$$12\frac{4}{5}=\frac{64}{5}:\frac{64}{5}\div4=\frac{16}{5}=3\frac{1}{5}.$$

In the second method we reduce $12\frac{4}{5}$ to the improper fraction $\frac{64}{5}$, and divide it as in Case 1, Art. 152.

Ex. 2. Divide $13\frac{1}{2}$ by 3. Ans. $4\frac{1}{2}$.

FIRST METHOD.

$$3\ )\ 1\ 3\tfrac{1}{2}$$
$$4\tfrac{1}{2}.$$

EXPLANATION.

In the first method we say "3 in 13 4 times, and 1 remains." Write 4, and reduce 1 to *halves*, making two halves; $\frac{2}{2} + \frac{1}{2} = \frac{3}{2}$: $\frac{3}{2} \div 3 = \frac{1}{2}$, making the whole quotient $4\frac{1}{2}$.

SECOND METHOD.

$$13\tfrac{1}{2} = \tfrac{27}{2} : \tfrac{27}{2} \div 3 = \tfrac{9}{2} = 4\tfrac{1}{2}.$$

In the second method we reduce the mixed number to an improper fraction, $\frac{27}{2}$, and divide it as a fraction.

Rule.—*Divide first the integral part. If there is a remainder, reduce it to the denominator of the fractional part, and add it to that part; then divide this sum.* Or

Reduce the mixed number to an improper fraction, and then divide by the given divisor.

EXAMPLES FOR PRACTICE.

Divide

3. $24\frac{18}{25}$ by 6. Ans. $4\frac{3}{25}$.
4. $36\frac{9}{10}$ by 9.
5. $25\frac{10}{17}$ by 5.
6. $21\frac{3}{4}$ by 3.
7. $32\frac{3}{5}$ by 6. Ans. $5\frac{13}{30}$
8. $33\frac{1}{3}$ by 5. Ans. $6\frac{2}{3}$.
9. $51\frac{2}{5}$ by 3.

Divide

10. $43\frac{3}{4}$ by 6. Ans. $7\frac{7}{24}$.
11. $62\frac{1}{3}$ by 8. Ans. $7\frac{19}{24}$.
12. $3\frac{2}{5}$ by 12. Ans. $\frac{17}{60}$.
13. $238\frac{1}{4}$ by 5.
14. $112\frac{7}{8}$ by 16.
15. $532\frac{5}{6}$ by 25.
16. $324\frac{3}{10}$ by 36. Ans. $9\frac{1}{120}$.

17. Divide $\$861\frac{3}{4}$ equally among 5 persons.

Ans. Each has $\$172\frac{7}{20}$.

18. If 40 horses cost $\$5131\frac{1}{4}$, what was the average cost per head?

19. A man expended $\$7935\frac{2}{7}$ in land, at \$43 per acre; how many acres did he buy? Ans. $184\frac{163}{301}$.

20. If 75 barrels of flour cost $\$783\frac{1}{3}$, what is the cost per barrel?

21. If 5 bushels of apples cost $\$2\frac{1}{3}$, what is the cost per bushel?

CASE III.

Art. 155. To divide a whole number by a fraction.

Ex. 1. Divide 16 by $\frac{2}{3}$. Ans. 24.

Process Indicated.

First.—$16 \div \frac{2}{3} = \frac{48}{3} \div \frac{2}{3} = 48 \div 2 = 24$.

Second.—$16 \div \frac{2}{3} = 3 \times (16 \div 2) = 3 \times 8 = 24$.

Explanation.—In the first method we bring the dividend 16 to *thirds* because the divisor is *thirds*. Then 2 thirds are contained in 48 thirds as many times as 2 is contained in 48, namely 24 times.

In the second method we reason thus:—If 16 is divided by 2, the quotient, 8, is *three times too small*, because it is not required to divide by the whole of 2, but by *one-third* of 2. Therefore, the true quotient is 3 times 8, or 24. This is expressed in the equation, $16 \div \frac{2}{3} = 16 \div \frac{1}{3}$ of 2.

Rule.—*Multiply the whole number by the denominator, and divide the product by the numerator.*

Examples for Practice.

Divide

2. 24 by $\frac{3}{4}$. Ans. 32.
3. 25 by $\frac{5}{6}$.
4. 2 by $\frac{7}{8}$. Ans. $2\frac{2}{7}$.
5. 40 by $\frac{3}{5}$. Ans. $66\frac{2}{3}$
6. 32 by $\frac{5}{6}$.
7. 41 by $\frac{2}{3}$. Ans. $61\frac{1}{2}$.

Divide

8. 63 by $\frac{7}{9}$.
9. 81 by $\frac{9}{10}$.
10. 120 by $\frac{7}{12}$.
11. 125 by $\frac{5}{8}$.
12. 132 by $\frac{11}{12}$.
13. 105 by $\frac{7}{8}$.

14. A lady expended \$7 for ribbon at the rate of \$$\frac{3}{5}$ per yard; how many yards did she buy? Ans. $11\frac{2}{3}$ yards.

15. A man paid \$3672 for $\frac{3}{4}$ of a farm; at the same rate, what is the whole farm worth? Ans. \$4896.

16. If potatoes are worth \$$\frac{5}{8}$ per bushel, how many bushels can be bought for \$72? Ans. $115\frac{1}{5}$.

17. B purchased $\frac{2}{5}$ of a mill for \$2127; at the same rate what is the mill worth? Ans. \$$5317\frac{1}{2}$.

Art. 156. When the dividend is the whole number 1, and the divisor is a fraction, the quotient is the *reciprocal* of the fraction. (See Art. 127.) On performing this division by the last rule, the following fact will appear:—

The reciprocal of a fraction is formed by interchanging its terms.

Illustration.—$1 \div \frac{4}{7} = \frac{7}{7} \div \frac{4}{7} = 7 \div 4 = \frac{7}{4}$. But $\frac{7}{4}$ is formed by interchanging the terms of $\frac{4}{7}$.

Examples for Practice.

Form the reciprocal of $\frac{1}{2} : \frac{1}{3} : \frac{1}{4} : \frac{1}{5} : \frac{2}{3} : \frac{3}{4} : \frac{2}{5} : \frac{5}{8} : \frac{4}{7} : \frac{7}{10} : \frac{1}{8} : \frac{3}{8} : \frac{4}{9} : \frac{7}{9} : \frac{5}{11} : \frac{7}{16} : \frac{9}{25} : \frac{6}{13} : \frac{13}{27} : \frac{25}{73} : \frac{101}{105} : \frac{16}{125} : \frac{125}{324}$.

Case IV.

Art. 157. To divide a fraction by a fraction.

Ex. 1. How many times is $\frac{2}{9}$ contained in $\frac{8}{9}$? Ans. 4.

Analysis.—Two *ninths* are contained in 8 *ninths* as many times as 2 is contained in 8, namely, 4 times.

Ex. 2. How many times is $\frac{2}{3}$ contained in $\frac{8}{9}$? Ans. $1\frac{1}{3}$.

First Analysis.—Two *thirds* are contained in $\frac{8}{9}$ three times as many times as 2 is contained in $\frac{8}{9}$. Now, 2 is contained $\frac{4}{9}$ times in $\frac{8}{9}$, and $\frac{2}{3}$ must be contained 3 times as many times, that is, 3 times $\frac{4}{9}$, or $\frac{12}{9} = 1\frac{3}{9} = 1\frac{1}{3}$ times. This result is best reached by multiplying the dividend by the reciprocal of the divisor thus:—

$$\text{Written Process.—}\frac{8}{9} \div \frac{2}{3} = \frac{\cancel{8}^{4}}{\cancel{9}_{3}} \times \frac{\cancel{3}}{\cancel{2}} = \frac{4}{3} = 1\tfrac{1}{3}.$$

Second Analysis.—Two-thirds are equal to $\frac{6}{9}$: $\frac{6}{9}$ is contained in $\frac{8}{9}$ as many times as 6 is contained in 8, namely, $1\frac{2}{6}$, or $1\frac{1}{3}$ times.

$$\text{Written Process.—}\frac{8}{9} \div \frac{2}{3} = \frac{8}{9} \div \frac{6}{9} = 8 \div 6 = 1\tfrac{2}{6} = 1\tfrac{1}{3}.$$

Third Analysis.—Two *thirds* are contained in $\frac{8}{9}$ three times as many times as 2 is contained in $\frac{8}{9}$. Now, 2 is contained $\frac{4}{9}$ times in $\frac{8}{9}$, which is found by dividing the numerator 8 by 2, according to Art. 153. Since $\frac{4}{9}$ is three times two small, the true quotient is 3 times $\frac{4}{9}$, namely, $\frac{4}{3}$, which is found by dividing the denominator, according to Art. 146.

$$\text{Written Process.—}\frac{8}{9} \div \frac{2}{3} = \frac{8 \div 2}{9 \div 3} = \frac{4}{3} = 1\tfrac{1}{3}.$$

Note.—By Art. 150 it appears that *the product of two fractions equals the product of the numerators over the product of the denominators.* By the analysis above it appears that *the quotient of one fraction by another equals the quotient of the numerators over the quotient of the denominators.*

Ex. 3. How many times is $\frac{1}{8}$ contained in $\frac{1}{2}$? Ans. 4.

Ex. 4. How many times is $\frac{1}{7}$ contained in $\frac{1}{6}$? Ans. $\frac{7}{6}=1\frac{1}{6}$.

PROCESS INDICATED.

For Ex. 3.—$\frac{1}{2} \div \frac{1}{8} = \frac{1}{2} \times \frac{8}{1} = \frac{8}{2} = 4$.

For Ex. 4.—$\frac{1}{6} \div \frac{1}{7} = \frac{1}{6} \times \frac{7}{1} = \frac{7}{6} = 1\frac{1}{6}$.

EXPLANATION.

On solving Examples 3 and 4 as Example 2 was solved, it appears that, *when both fractions have 1 for a numerator, the quotient is equal to the denominator of the divisor over that of the dividend.*

Ex. 5. Divide $\frac{3}{4}$ of $\frac{4}{5}$ by $\frac{5}{6}$ of $\frac{6}{7}$. Ans. $\frac{21}{25}$.

PROCESS INDICATED.

$$\frac{3}{\cancel{4}} \times \frac{\cancel{4}}{5} \times \frac{\cancel{6}}{5} \times \frac{7}{\cancel{6}} = \frac{3 \times 7}{5 \times 5} = \frac{21}{25}.$$

EXPLANATION.

This is solved like the first method of Example 2, namely by multiplying the dividend by the reciprocal of every term of the divisor.

Rule.—*Multiply the dividend by the reciprocal of the divisor.*

NOTE 1.—Reduce mixed numbers first to fractions.

NOTE 2.—If the fractions have a common denominator, the quotient is the quotient of the numerator of the dividend by that of the divisor.

NOTE 3.—If the fractions have a common numerator, the quotient is the quotient of the denominator of the divisor by that of the dividend.

NOTE 4.—If each term of the divisor exactly measures the corresponding term of the dividend, the quotient may be found by placing the quotient of the numerators over the quotient of the denominators.

NOTE 5.—After writing the reciprocal of the divisor as a multiplier, abbreviate by canceling, if possible.

EXAMPLES FOR PRACTICE.

Divide

6. $\frac{8}{11}$ by $\frac{4}{11}$. Ans. 2.
7. $\frac{12}{19}$ by $\frac{3}{19}$.
8. $\frac{15}{23}$ by $\frac{5}{23}$.
9. $\frac{5}{6}$ by $\frac{1}{3}$. Ans. $2\frac{1}{2}$.
10. $\frac{2}{7}$ by $\frac{3}{7}$. Ans. $\frac{2}{3}$.
11. $\frac{6}{6}$ by $\frac{7}{8}$. Ans. $\frac{2}{2}$
12. $\frac{7}{8}$ by $\frac{5}{6}$. Ans. $1\frac{1}{20}$.
13. $\frac{3}{10}$ by $\frac{4}{5}$.
14. $\frac{4}{5}$ by $\frac{8}{10}$.

Divide

15. $\frac{3}{7}$ by $\frac{5}{8}$. Ans. $\frac{24}{35}$.
16. $\frac{5}{8}$ by $\frac{3}{7}$.
17. $\frac{1}{2}$ by $\frac{7}{8}$.
18. $\frac{7}{8}$ by $\frac{1}{2}$.
19. $\frac{9}{16}$ by $\frac{7}{8}$.
20. $\frac{16}{25}$ by $\frac{4}{5}$.
21. $\frac{18}{19}$ by $\frac{12}{95}$. Ans. $7\frac{1}{2}$.
22. $\frac{3}{4}$ of $\frac{5}{6}$ by $\frac{7}{8}$. Ans. $\frac{5}{7}$.
23. $\frac{3}{4}$ by $\frac{5}{6}$ of $\frac{7}{8}$. Ans. $1\frac{1}{35}$.

24. $\frac{5}{8}$ of $\frac{8}{9}$ by $\frac{2}{5}$ of $\frac{5}{6}$. Ans. $1\frac{2}{3}$.

25. $\frac{2}{3}$ of $\frac{3}{7}$ by $\frac{2}{7}$ of $\frac{9}{10}$. Ans. $1\frac{1}{9}$.

26. $\frac{2}{5}$ of $\frac{1}{4}$ by $\frac{3}{4}$ of $\frac{5}{6}$. Ans. $\frac{4}{25}$.

27. $\frac{5}{6}$ of $\frac{3}{8}$ by $\frac{3}{5}$ of $\frac{5}{9}$.

28. $\frac{18}{51}$ of $\frac{34}{39}$ by $\frac{25}{42}$ of $\frac{28}{55}$.

29. $\frac{5}{6}$ of $8\frac{1}{3}$ by $\frac{4}{5}$ of $6\frac{2}{3}$. Ans. $5\frac{5}{24}$.

30. $\frac{3}{5}$ of $7\frac{1}{2}$ by $\frac{3}{4}$ of 10. Ans. $\frac{3}{5}$.

31. $\frac{2}{3} \times \frac{3}{4} \times \frac{8}{9}$ by $\frac{5}{6}$ of $\frac{1}{2}$ of $2\frac{2}{5}$.

32. $\frac{4}{5} \times \frac{5}{8} \times 7\frac{1}{3}$ by $\frac{7}{10} \times \frac{11}{14} \times \frac{5}{6}$ of $\frac{2}{3}$.

33. If $\frac{8}{15}$ of an acre of land sells for $\$53\frac{1}{3}$, at what price per acre does it sell? Ans. \$100.

34. If $3\frac{2}{3}$ yards of ribbon cost $\$11\frac{2}{3}$, what is the cost per yard? Ans. $\$3\frac{2}{11}$.

35. If $8\frac{5}{8}$ yards of lace cost $\$5\frac{3}{4}$, what is the cost per yard?

36. If $7\frac{1}{4}$ gallons of syrup cost $\$5\frac{4}{5}$, what is the price per gallon?

37. If 1 man can complete a job of work in $6\frac{7}{9}$ hours, how many hours would it take 5 men to do the same?

38. At $\$\frac{1}{6}$ per pound, how many pounds of cheese can C buy for $\$3\frac{1}{3}$?

39. At $\frac{4}{5}$ of a cent each, how many apples can be bought for $9\frac{3}{5}$ cents?

40. If I pay $3\frac{1}{3}$ cents for riding 1 mile, how far can I ride for $116\frac{2}{3}$ cents?

41. At $\$2\frac{4}{5}$ a day, how many days must a man work for $\$74\frac{1}{5}$?

42. If a man can mow a field in 7 days, what part of it can he mow in 1 day?

Art. 158. To find the value of a complex fraction.

Ex. 1. What is the value of $\dfrac{3\frac{3}{4}}{\frac{5}{8}}$? Ans. 6.

PROCESS INDICATED.

$$\frac{3\frac{3}{4}}{\frac{5}{8}} = 3\tfrac{3}{4} \div \tfrac{5}{8} = \tfrac{15}{4} \div \tfrac{5}{8} = \tfrac{15}{4} \times \tfrac{8}{5} = 6.$$

EXPLANATION.

Since the value of a fraction is the quotient of its numerator by its denominator, the value of this fraction is $3\frac{3}{4} \div \frac{5}{8} = 6$.

Rule.—*Divide the numerator by the denominator.*

NOTE.—The solution of a complex fraction is only an application of some one of the preceding cases of the divison of fractions.

EXAMPLES FOR PRACTICE.

What is the value

2. Of $\dfrac{\frac{2}{3}}{\frac{3}{4}}$? Ans. $\frac{8}{9}$.

3. Of $\dfrac{\frac{5}{6}}{\frac{2}{5}}$? Ans. $2\frac{1}{12}$.

4. Of $\dfrac{\frac{3}{5}}{7}$? Ans. $\frac{3}{35}$.

5. Of $\dfrac{10}{\frac{5}{8}}$? Ans. 16.

6. Of $\dfrac{2\frac{3}{4}}{11}$? Ans. $\frac{1}{4}$.

7. Of $\dfrac{7\frac{2}{3}}{12}$?

8. Of $\dfrac{\frac{5}{9}}{4\frac{1}{3}}$?

9. Of $\dfrac{4\frac{1}{3}}{\frac{5}{9}}$

10. Of $\dfrac{3\frac{2}{5}}{5\frac{1}{6}}$?

11. Of $\dfrac{18\frac{3}{4}}{\frac{3}{8}}$?

12. $\dfrac{\frac{5}{8}}{\frac{5}{6}} + \dfrac{\frac{6}{7}}{12} + \dfrac{16\frac{2}{3}}{6\frac{1}{4}} = ?$ Ans. $3\frac{41}{84}$.

13. $\dfrac{12\frac{1}{2}}{8\frac{1}{3}} - \dfrac{8\frac{1}{3}}{12\frac{1}{2}} = ?$ Ans. $\frac{5}{6}$.

14. $\dfrac{2\frac{1}{3}}{3\frac{1}{2}} \times \dfrac{\frac{3}{8}}{\frac{6}{7}} \times \dfrac{\frac{5}{6}}{10} \times \dfrac{10}{\frac{5}{6}} = ?$ Ans. $\frac{7}{24}$.

15. $\dfrac{10}{\frac{3}{4}} \div \dfrac{\frac{5}{6}}{1\frac{1}{2}} = ?$ Ans. 24.

Continued Fractions.

Art. 159. **A continued fraction** is a complex fraction whose numerator is 1, and whose denominator is a whole number, and a fraction whose numerator is 1, and whose denominator is similar to the preceding one, and so on. Thus, if we divide both terms of $\frac{13}{41}$ by 13, we obtain

$$\frac{1}{3\frac{2}{13}}$$

If we divide both terms of $\frac{2}{13}$ by 2, we obtain

$$\cfrac{1}{3+\cfrac{1}{6+\frac{1}{2}}}$$

which is a continued fraction, equal to $\frac{13}{41}$, because dividing both terms by the same number does alter the value of the fraction.

The **terms** of a continued fraction are the simple fractions which form the continued fraction. Thus, the terms of the above continued fraction are $\frac{1}{3}$, $\frac{1}{6}$, and $\frac{1}{2}$.

Art. 160. To reduce a common fraction to a continued fraction.

Rule.—*Divide both terms of the given fraction by the numerator. If the fraction in the denominator thus formed has not 1 for a numerator, divide both terms of it by its numerator, and so on, till the last fraction has 1 for a numerator.*

Note.—If the given fraction is improper, find its value, then reduce its fractional part to a continued fraction.

Examples for Practice.

1. Reduce $\frac{35}{158}$ to a continued fraction.

Ans. The terms are $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{5}$, $\frac{1}{3}$.

2. Reduce $\frac{68}{157}$ to a continued fraction.

Ans. The terms are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$.

Reduce to a continued fraction.

3. $\frac{24}{83}$. Ans. The terms are $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{5}$, $\frac{1}{2}$.
4. $\frac{55}{192}$. Ans. The terms are $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{27}$.
5. $\frac{47}{303}$. Ans. The terms are $\frac{1}{6}$, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{5}$.
6. $\frac{227}{953}$. Ans. The terms are $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{22}$, $\frac{1}{2}$.
7. $\frac{99}{823}$. Ans. The terms are $\frac{1}{8}$, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{6}$.
8. $\frac{71}{365}$. Ans. The terms are $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{10}$.
9. $\frac{109}{360}$. Ans. The terms are $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{3}$.
10. $\frac{44}{461}$. Ans. The terms are $\frac{1}{10}$, $\frac{1}{2}$, $\frac{1}{10}$, $\frac{1}{2}$.

Art. 161. To approximate to the value of a continued fraction.

Ex. 1. Find the successive approximations to the value of $\frac{35}{156}$.

APPROXIMATIONS.

FIRST.—$\frac{1}{4}$; too large, because the denominator is more than 4.

SECOND. $\dfrac{1}{4\frac{1}{2}} = \frac{2}{9}$; too small, because $4\frac{1}{2}$ is too large.

THIRD. $\dfrac{1}{4\cfrac{1}{2\frac{1}{5}}} = \dfrac{1}{4\cfrac{1}{\frac{11}{5}}} = \dfrac{1}{4\frac{5}{11}} = \frac{11}{49}$; too large.

FOURTH. $\dfrac{1}{4\cfrac{1}{2\cfrac{1}{5\frac{1}{3}}}} = \dfrac{1}{4\cfrac{1}{2\frac{3}{16}}} = \dfrac{1}{4\frac{16}{35}} = \dfrac{1}{\frac{156}{35}} = \frac{35}{156}$.

Rule.—*Take the first term for the first approximation. Find successively the values of the first part of the continued fraction, including one more term at each trial.*

EXAMPLES FOR PRACTICE.

Find the successive approximations to the value

2. Of $\frac{68}{157}$. Ans. $\frac{1}{2}$, $\frac{3}{7}$, $\frac{13}{30}$, $\frac{68}{157}$.
3. Of $\frac{24}{83}$. Ans. $\frac{1}{3}$, $\frac{2}{7}$, $\frac{11}{38}$, $\frac{24}{83}$.
4. Of $\frac{55}{192}$.
5. Of $\frac{47}{103}$. Ans. $\frac{1}{6}$, $\frac{2}{13}$, $\frac{9}{58}$, $\frac{47}{303}$.
6. Of $\frac{227}{958}$.
7. Of $\frac{99}{823}$.
8. Of $\frac{71}{365}$.
9. Of $\frac{109}{360}$.
10. Of $\frac{44}{461}$.

GREATEST COMMON DIVISOR OF FRACTIONS.

Art. 162. A common divisor of two or more fractions is a number which is contained in each of them a whole number of times. Thus, $\frac{2}{11}$ is a divisor common to $\frac{6}{11}$ and $\frac{10}{11}$, being contained in $\frac{6}{11}$ three times, and in $\frac{10}{11}$ five times.

The **greatest common divisor** of two or more fractions is the greatest number which is contained in each of them a whole number of times. Thus, $\frac{4}{9}$ is the greatest common divisor of $\frac{4}{9}$ and $\frac{8}{9}$.

Art. 163. To find the greatest common divisor of two or more fractions.

Ex. 1. Find the greatest common divisor of $\frac{4}{5}$ and $\frac{8}{9}$.

Ans. $\frac{4}{45}$.

METHODS INDICATED.

	First.	Second.
$\frac{4}{5}=\frac{36}{45}$.	$36)40(1$	$\frac{36}{45})\frac{40}{45}(1$
$\frac{8}{9}=\frac{40}{45}$.	36	$\frac{36}{45}$
	—	—
	$4)36(9$.	$\frac{4}{45})\frac{36}{45}(9$.

EXPLANATION.

Since $\frac{4}{5}=\frac{36}{45}$, and $\frac{8}{9}=\frac{40}{45}$, and since the greatest common divisor of 36 and 40 is 4, the greatest common divisor of $\frac{36}{45}$ and $\frac{40}{45}$ must be $\frac{4}{45}$.

Since 45 is the least common denominator of $\frac{4}{5}$ and $\frac{8}{9}$, and since, if the common denominator is the least possible, the value of the common divisor, $\frac{4}{45}$,

must be the greatest possible, $\frac{4}{45}$ must be the greatest common divisor of $\frac{4}{5}$ and $\frac{8}{9}$. In these processes it appears that the greatest common divisor, $\frac{4}{45}$, is composed of the greatest common divisor, 4, of the numerators, 4 and 8, of the given fractions, placed over the least common multiple, 45, of their denominators 5 and 9.

Rule.—*Write the greatest common divisor of the given numerators over the least common multiple of the given denominators.*

NOTE 1.—The fractions may first be reduced to their lowest terms, or the greatest common divisor, when found, may be reduced to its lowest terms.

NOTE 2.—The greatest common divisor of fractions may be found precisely as that of integers is found, namely, by dividing one by another, and the divisor by the remainder, till there is no remainder; (See Art. 110;) but the method stated in the rule is the simplest and best.

EXAMPLES FOR PRACTICE.

Find the greatest common divisor

2. Of $\frac{6}{7}$ and $\frac{9}{10}$. Ans. $\frac{3}{70}$.
3. Of $\frac{4}{9}$ and $\frac{2}{3}$. Ans. $\frac{2}{3}$.
4. Of $\frac{3}{4}$ and $\frac{2}{5}$. Ans. $\frac{1}{20}$.
5. Of $\frac{6}{11}$ and $\frac{12}{17}$.
6. Of $8\frac{1}{3}$ and $12\frac{1}{2}$. Ans. $4\frac{1}{6}$.
7. Of $7\frac{1}{2}$ and $22\frac{1}{2}$. Ans. $7\frac{1}{2}$.
8. Of $11\frac{3}{7}$ and $13\frac{5}{7}$. Ans. $2\frac{2}{7}$.
9. Of $89\frac{5}{6}$ and $102\frac{2}{3}$. Ans. $12\frac{5}{6}$.
10. Of $\frac{3}{5}$ and $\frac{5}{8}$.
11. Of $\frac{4}{9}$ and $\frac{8}{11}$.
12. Of $\frac{16}{21}$ and $\frac{24}{49}$.
13. Of $\frac{20}{33}$ and $\frac{30}{55}$.
14. Of $8\frac{1}{3}$ and 10. Ans. $1\frac{2}{3}$.
15. Of $18\frac{3}{4}$ and 25. Ans. $6\frac{1}{4}$.
16. $\dfrac{2\frac{2}{3}}{\frac{3}{4}}$ and $\dfrac{1\frac{3}{4}}{2\frac{1}{3}}$. Ans. $\frac{1}{36}$.

LEAST COMMON MULTIPLE OF FRACTIONS.

Art. 164. **A common multiple** of two or more fractions is a number which contains each of them a whole number of times.

The **least common multiple** of two or more fractions is the least number which contains each of them a whole number of times.

Art. 165. To find the least common multiple of two or more fractions.

Ex. 1. Find the least common multiple of $\frac{3}{4}$, $\frac{5}{8}$, and $\frac{15}{16}$.
Ans. $3\frac{3}{4}$.

FIRST METHOD.

$\frac{3}{4} = \frac{12}{16} : \frac{5}{8} = \frac{10}{16}$:

$$\begin{array}{r|ccc} 5 & \frac{12}{16} & \frac{10}{16} & \frac{15}{16} \\ \hline 3 & \frac{12}{16} & \frac{2}{16} & \frac{3}{16} \\ \hline 2 & \frac{4}{16} & \frac{2}{16} & \frac{1}{16} \\ \hline \frac{1}{16} & \frac{2}{16} & \frac{1}{16} & \frac{1}{16} \\ \hline & 2 & 1 & 1 \end{array}$$

$5 \times 3 \times 2 \times \frac{1}{16} \times 2 = 3\frac{3}{4}$.

EXPLANATION.

In the first method we proceed with the fractions exactly as with whole numbers, after having reduced them to a common denominator. The analysis is briefly as follows:—

The least common multiple of 12, 10, and 15 is 60: therefore the least common multiple of $\frac{12}{16}$, $\frac{10}{16}$, and $\frac{15}{16}$ is $\frac{60}{16}$, or $3\frac{3}{4}$.

SECOND METHOD.

$$\begin{array}{r|ccc} 5 & 3 & 5 & 15 \\ \hline 3 & 3 & 1 & 3 \\ \hline & 1 & 1 & 1 \end{array}$$

$5 \times 3 = 15$, l. c. m. of num.
$4 =$ gr. com. div. of denom.
Ans. $= \frac{15}{4} = 3\frac{3}{4}$.

In the second method the least common multiple of the numerators of the fractions is found, namely, 15, and the greatest common divisor of the denominators, namely, 4; and the least common multiple of the fractions must be 15 over 4, or $3\frac{3}{4}$.

DEMONSTRATION.—The least number which will contain 3, 5, and 15 a whole number of times will contain $\frac{3}{4}$, $\frac{5}{8}$, and $\frac{15}{16}$, respectively, 4, 8 and 16 times as many times, that is, once 4, twice 4, and four times 4 times as many times, respectively. Therefore, 15, the least common multiple of 3, 5, and 15, is 4 times as large as the least common multiple of $\frac{3}{4}$, $\frac{5}{8}$, and $\frac{15}{16}$ is $\frac{15}{4} = 3\frac{3}{4}$.

Rules.—I. *Reduce the fractions, if necessary, to their least common denominator, and proceed with them as in finding the least common multiple of whole numbers.* Or

II. *Reduce the fractions, if necessary, to their lowest terms; then divide the least common multiple of the numerators by the greatest common divisor of the denominators.*

NOTE.—The least *whole number* that will contain two or more fractions, in their lowest terms, a whole number of times, is the *least common multiple of their numerators.*

Examples for Practice.

Find the least common multiple

2. Of $\frac{3}{4}$ and $\frac{5}{8}$. Ans. $3\frac{3}{4}$.
3. Of $\frac{5}{6}$ and $\frac{3}{7}$. Ans. 15.
4. Of $\frac{7}{12}$ and $\frac{13}{18}$.
5. Of $\frac{9}{10}$ and $\frac{7}{15}$.
6. Of $\frac{12}{35}$ and $\frac{16}{21}$. Ans. $6\frac{6}{7}$.
7. Of $\frac{14}{25}$ and $\frac{21}{55}$.
8. Of $2\frac{1}{3}$ and $3\frac{1}{2}$. Ans. 7.
9. Of $6\frac{1}{4}$ and $8\frac{1}{3}$.
10. Of $12\frac{1}{2}$ and $31\frac{1}{4}$.
11. Of $19\frac{5}{6}$ and $22\frac{2}{3}$.
12. Of $16\frac{3}{7}$ and 23. Ans. $3\frac{2}{7}$.
13. Of $\frac{8}{9}$, $\frac{5}{6}$, $\frac{11}{12}$, $\frac{4}{15}$.
14. Of $\frac{3}{4}$, $\frac{7}{8}$, $\frac{11}{12}$.
15. Of $3\frac{3}{8}$, $5\frac{3}{4}$, $\frac{3}{8}$.
16. Of $\dfrac{\frac{7}{12} \text{ of } 2\frac{5}{8}}{\frac{5}{8} \text{ of } 8\frac{3}{4}}$, $\dfrac{\frac{2}{7}}{\frac{5}{6}}$. Ans. $16\frac{4}{5}$.
17. Of $\dfrac{\frac{3}{8}}{\frac{4}{15}}$ and $\dfrac{2\frac{5}{6}}{5\frac{1}{3}}$. Ans. $9\frac{27}{32}$.

Miscellaneous Exercises in Common Fractions.

1. Reduce $123\frac{7}{12}$ to an improper fraction.
2. Reduce $\frac{2132}{25}$ to a mixed number.
3. Reduce $\frac{1107}{1722}$ to its lowest terms.
4. $\frac{5}{8}$ of $\frac{6}{7}$ of $\frac{9}{14}$ of $\frac{16}{27}$ of $\frac{21}{25}$ of $\frac{20}{9}$ = ?
5. Reduce $\frac{5}{6}$, $\frac{4}{9}$, $\frac{11}{18}$, $\frac{7}{12}$, $3\frac{1}{3}$, and 4 to their least common denominator.
6. What is the sum of $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{12}$, and $\frac{13}{18}$?
7. $\frac{3}{4}$ of $\frac{8}{9}$ of $11\frac{2}{5} + \frac{3}{4}$ of $24 + 6\frac{1}{2} + 8\frac{1}{3}$ = ?
8. $\frac{5}{6}$ of $3\frac{3}{5} - \frac{2}{3}$ of $1\frac{2}{3}$ = ?
9. What is the square of $4\frac{1}{2}$?
10. The sum of two numbers is $\dfrac{4\frac{1}{2}}{7\frac{1}{3}}$ and the less is $\dfrac{\frac{3}{44}}{\frac{4}{7}}$, what is the greater?
11. The less of two numbers is $5\frac{7}{8}$, and their difference $\dfrac{\frac{5}{6}}{2\frac{2}{9}}$; what is the greater?
12. What is the sum of $4\frac{3}{7}$ + its reciprocal?
13. From $9\frac{3}{5}$ subtract its reciprocal.
14. Multiply $\frac{6}{7}$ by its reciprocal.

15. What number multiplied by $\frac{5}{6}$ of $\dfrac{2\frac{1}{2}}{3\frac{1}{3}}$ will produce $\frac{7}{11}$?

16. What, divided by $1\frac{2}{3}$ gives $21\frac{1}{2}$?

17. A has $5\frac{3}{4}$ times $\$8\frac{1}{3}$, B has $1\frac{2}{3}$ times $\$10\frac{2}{5}$; how much have they both?

18. Divide $3\frac{3}{4}$ times $2\frac{8}{15}$ by $\overline{(\frac{3}{5} \times 4\frac{1}{6}) - (\frac{2}{3} \times \frac{4}{5})}$.

19. Divide $2\frac{1}{3}$ by the reciprocal of $1\frac{2}{5}$.

20. From the square of $1\frac{2}{5}$ subtract the cube of $1\frac{1}{10}$.

21. A has \$7225, which is $1\frac{1}{4}$ times as much as B has; how much has B?

22. What number diminished by the difference between $\frac{3}{5}$ and $\frac{7}{8}$ of itself, leaves a remainder 148?

23. A and B together bought 680 sheep. A paid \$9 as often as B paid \$8; what number ought each to receive?

Ans. A 360, and B 320.

24. A and B own 288 horses; how many has each, if B has $\frac{3}{5}$ as many as A? Ans. A 180, B 108.

25. If $2\frac{1}{2}$ barrels of flour cost $\$26\frac{1}{4}$, how much will $8\frac{1}{3}$ barrels cost?

26. How many pounds of butter at $\$\frac{3}{16}$ per pound will pay for 48 bushels of corn at $\$\frac{3}{4}$ per bushel?

27. What number increased by $\frac{3}{5}$ of itself equals 32?

Ans. 20.

28. What number diminished by $\frac{3}{5}$ of itself equals 32?

Ans. 80.

29. Divide \$2500 between two persons, so that one shall have $\frac{9}{16}$ as much as the other.

30. A owned $\frac{5}{6}$ of a mill, and sold $\frac{3}{7}$ of his interest; what part of the mill did he sell, and what part has he left?

31. If 35 is $\frac{5}{8}$ of some number, what is $\frac{4}{7}$ of the same number?

32. What is the quotient of the sum of $4\frac{2}{5}$ and $2\frac{2}{3}$ by their difference?

33. What two numbers between 32 and 480, have the former for their greatest common divisor and the latter for their least common multiple? (See Art. 115.)

Ans. 96 and 160.

34. At $\$\frac{8}{9}$ per yard, how many yards of flannel can be bought for $\$8\frac{3}{4}$?

35. How many bushels of wheat at $\$1\frac{5}{8}$ per bushel, will pay for $\frac{5}{6}$ of a barrel of flour at $\$8\frac{2}{3}$ per barrel?

36. If a person can do a job of work in 24 days, what part of it can he do in 16 days?

37. If a person going $4\frac{1}{3}$ miles per hour, performs a journey in $10\frac{3}{4}$ hours, how many hours would it take him, if he traveled $3\frac{1}{2}$ miles per hour?

38. Bought $12\frac{1}{2}$ barrels of flour at $\$10\frac{1}{2}$ per barrel, how many barrels more could I have bought for the same money, if I had paid $\$2\frac{2}{3}$ less per barrel? Ans. $4\frac{12}{47}$ barrels.

39. Bought $31\frac{1}{2}$ bushels of wheat at $\$1\frac{3}{5}$ per bushel, how many bushels less would I have bought for the same money, if I had paid $\$\frac{7}{10}$ more per bushel? Ans. $9\frac{27}{46}$ bushels.

40. What is the greatest breadth of carpet that would cover the floors of three rooms with whole breadths, the rooms being, respectively, $14\frac{2}{5}$ feet, $16\frac{4}{5}$ feet, and $19\frac{1}{5}$ feet broad? (See Arts. 162, and 163). Ans. $2\frac{2}{5}$ feet.

41. A miller has $53\frac{1}{3}$ bushels of oats, $66\frac{2}{3}$ bushels of corn, and $85\frac{1}{3}$ bushels of wheat, which he wishes to put in sacks of equal size without mixing. What is the capacity of each sack, and the number of sacks?

Ans. $2\frac{2}{3}$ bushels, and 77 sacks.

42. What is the least number of cents that will buy either a whole number of oranges at $6\frac{2}{9}$ cents each, or of lemons at $5\frac{3}{5}$ cents each? (See Arts. 164, and 165.) Ans. 56.

43. What is the least sum of money for which I can buy a number of ducks, at $\$\frac{7}{8}$ each, a number of geese, at $\$1\frac{7}{12}$ each, or a number of turkeys at $\$1\frac{3}{4}$ each, and how many of each *could be bought* for the same sum?

44. If A can dig a ditch in 6 days, and B can dig it in 9 days, what part of it can they both dig in 1 day? Ans. $\frac{5}{18}$.

45. If A can mow a field in 10 days, B in 12 days, and C in 15 days, in how many days can they all mow it?

Ans. 4 days.

46. If A can build a wall in 20 days, and A and B in 8 days, how long will it take B, alone, to build it?

Ans. $13\frac{1}{3}$ days.

47. A, B, and C can reap a field in 10 days, A and C in 15 days, and B and C in 20 days; in how many days could each alone reap it?

Ans. A in 20 days, B in 30 days, and C in 60 days.

48. C has 341 sheep in two fields. How many are in each field, provided the first contains $\frac{4}{7}$ as many as the second?

49. A man spent $372, and saved the remainder of his salary which was $\frac{17}{23}$ of it; what was his salary?

Ans. $1426.

50. A farmer has $\frac{3}{10}$ of his farm in grain, $\frac{3}{8}$ in grass, and the remainder, 91 acres in timber. How many acres in his farm, and how many in grain and grass respectively?

51. A tree whose length was 133 feet, was broken into two pieces by falling; $\frac{3}{5}$ of the length of the longer piece equaled $\frac{2}{3}$ of the length of the shorter. What was the length of each piece? Ans. 70 feet, and 63 feet.

52. Divide $2125 between A and B, so that $\frac{2}{3}$ of A's part shall equal $\frac{3}{4}$ of B's.

53. A owns $\frac{3}{13}$ of an oil well, valued at $27547; B owns $\frac{2}{5}$ of the remainder; C owns $\frac{6}{13}$ as much as A and B; and D owns the remainder. How much does each own?

Ans. A owns $6357, B $8476, C $6846, and D $5868.

54. A man dying left an estate to be divided among his four sons, A, B, C, D. A was to have $\frac{1}{5}$ of it; B $\frac{9}{40}$ of it; C $\frac{1}{4}$ of it; and D the remainder, which is $12480. What was the estate, and what did each receive?

55. If $\frac{9}{16}$ of a farm is cleared, and $\frac{7}{16}$ of it is timber land,

how many acres are in the farm, provided there are 32 acres more cleared than timber land, and how many acres are there of each kind?

56. C's farm cost $3500, and $\frac{3}{5}$ of its cost is $\frac{7}{6}$ of the cost of his house; what was the cost of the house?

57. A, B, and C bought a drove of sheep. A received $\frac{2}{5}$ of them; B $\frac{1}{3}$ of them; and C the remainder. If A had 104 more than B, how many were in the drove, and how many did each receive? Ans. 1560: A 624, B 520, C 416.

SYNOPSIS OF FRACTIONS.

KINDS.	DEFINITIONS.	KINDS.	OPERATIONS.
		Proper.	
		Improper.	Reduction.
	Notation.	Simple.	Addition.
	Numeration.	Complex.	Subtraction.
Common.	Signification.	Continued.	Multiplication.
	Value.	Single.	Division.
	Variation.	Compound.	G. Com. Div.
		Mixed.	L. Com. Mult.
Decimal.		Reciprocal.	

CHAPTER X.

DECIMAL FRACTIONS.

Art. 166. **A decimal fraction** is a fraction whose denominator is a power of 10. (See Art. 44). Thus, $\frac{7}{10}$, $\frac{35}{100}$, $\frac{629}{1000}$, $\frac{1234}{10000}$, are decimal fractions.

NOTE.—Decimal fractions are only a variety of common fractions.

NOTATION OF DECIMAL FRACTIONS.

Art. 167. Decimal fractions *may* be written like common fractions, but the fact that their denominators are decimal permits the fractions to be written without their denominators, and this is usually done.

As, in integers, every place, reckoning from left to right toward units, is one-tenth of the value of the preceding place, so places to the right of units may express decimal fractions, each one-tenth of the preceding.

ILLUSTRATION.

The unit is	1.
$\frac{1}{10}$ of a unit may be written	.1
$\frac{1}{10}$ of $\frac{1}{10}$, or $\frac{1}{100}$, may be written	.01
$\frac{1}{10}$ of $\frac{1}{100}$, or $\frac{1}{1000}$, may be written	.001
$\frac{1}{10}$ of $\frac{1}{1000}$, or $\frac{1}{10000}$, may be written	.0001
$\frac{1}{10}$ of $\frac{1}{10000}$, or $\frac{1}{100000}$, may be written	.00001
$\frac{1}{10}$ of $\frac{1}{100000}$, or $\frac{1}{1000000}$, may be written	.000001

Hence, *the numerator of any decimal fraction will, without its denominator, express the value of the fraction, if it is written as many places to the right of the place of units as the fraction has ciphers in its denominator.*

Art. 168. When decimal fractions are written without their denominators, they are usually called *decimals.*

The **decimal point,** or **separatrix,** is a period placed between the units' place and a decimal. Thus, in the mixed number $2\frac{3}{10}$, if the fraction is written as a decimal, a period is placed between 2 and 3, making the number 2.3.

NUMERATION OF DECIMALS.

Art. 169. Proceeding from the decimal point towards the right, the names of the places are *tenths, hundredths, thousandths, ten-thousandths, hundred-thousandths, millionths, ten-millionths, hundred-millionths, billionths, ten-billionths, &c.*

NUMERATION TABLE OF DECIMALS.

Decimal Point.	Tenths.	Hundredths.	Thousandths.	Ten-thousandths.	Hundred-thousandths.	Millionths.	Ten-millionths.	Hundred-millionths.	Billionths.	
.	9									is read 9 *tenths.*
.	8	9								is read 89 *hundredths.*
.	7	8	9							is read 789 *thousandths.*
.	6	7	8	9						is read 6789 *ten-thousandths.*
.	5	6	7	8	9					is read 56789 *hundred-thousandths.*
.	4	5	6	7	8	9				is read 456789 *millionths.*
.	3	4	5	6	7	8	9			is read 3456789 *ten-millionths.*
.	2	3	4	5	6	7	8	9		is read 23456789 *hundred-millionths.*
.	1	2	3	4	5	6	7	8	9	is read 123456789 *billionths.*

Art. 170. To read decimals.

Rule.—*Reckon from the decimal point, and find the name of the place of the right-hand figure; then read the figures as a whole number of that name.*

NOTE 1.—To avoid misunderstanding on the part of the hearer, it is best to introduce the reading of a pure decimal with the words *the decimal,* and the reading of the decimal part of a mixed number with the words *with the decimal,* or by using the word *and* between the whole number and the decimal. Thus, in pronouncing .1059, by saying "*The decimal one thousand and fifty-nine ten-thousandths,*" we avoid being understood to mean 1000.0059; and, in pronouncing 100.003, by saying "*One hundred, with the decimal three thousandths,*" we avoid being understood to mean .103.

NOTE 2.—In reckoning a decimal according to the rule, we numerate toward the right, to find the name of the right-hand place, and toward the left, to be able to read it as a whole number.

EXAMPLES FOR PRACTICE.

1. Read .75.
2. .078.
3. .0109.
4. 4.26.
5. 10.1.
6. 23.23.
7. 7.204.
8. .032.
9. .008.
10. .404.
11. 16.0625.
12. .0407.
13. 100.01.
14. 45.0105.
15. 212.212.
16. 305.0035.
17. 123.203402.
18. 1004.00005.
19. 2.01030407.
20. .46070032.
21. 1000.001.
22. 5005.005.
23. 15.00032.
24. 6.000207.
25. .0002006.
26. .41532754.
27. $.6\frac{1}{3}$.
28. $.7\frac{3}{4}$.
29. $.08\frac{1}{3}$.
30. $.00156\frac{1}{4}$.
31. $45.166\frac{2}{3}$.
32. $44.4444\frac{4}{9}$.
33. $175.148\frac{2}{7}$.

Art. 171. To write decimals.

Rule.—*Write the decimal as a whole number; then make its right-hand figure occupy the place which names the decimal. Place the decimal point at the left of this arrangement.*

ILLUSTRATION.—To write the decimal *five hundred seven hundred-thousandths,* we first write 507: then the last figure, 7, must be made to occupy the place of hundred-thousandths, which is the fifth place from units. Hence, we prefix two ciphers and the decimal point, thus, .00507.

Examples for Practice.

Write in figures

1. Sixty-six hundredths.
2. Forty-seven thousandths.
3. One hundred three ten-thousandths.
4. Twenty-five, and twenty-one hundred-thousandths.
5. Ten, and two hundred seven millionths.
6. Three hundred forty-three ten-millionths.
7. Ten thousand, and 7 ten-thousandths.
8. Sixty, and 16 hundred-millionths.
9. One hundred six, and 106 billionths.
10. Six thousand four ten-billionths.

Express the following fractions and mixed numbers decimally:—

11. $\frac{7}{10}$. Ans. .7.
12. $\frac{65}{100}$.
13. $\frac{43}{1000}$.
14. $\frac{219}{100000}$. Ans. .00219.
15. $\frac{613}{10000}$.
16. $\frac{53}{1000000}$.
17. $\frac{20001}{1000000}$.
18. $\frac{803}{1000000}$.
19. $\frac{71}{10000000}$.
20. $\frac{43}{10}$. Ans. 4.3.
21. $\frac{711}{100}$. Ans. 7.11.
22. $6\frac{41}{100}$. Ans. 6.41.
23. $104\frac{104}{1000}$.
24. $70\frac{23}{10000}$.
25. $400\frac{1001}{1000000}$.
26. $403\frac{403}{100000}$.
27. $45\frac{7}{10000000}$.
28. $300000\frac{3}{100000}$.

Art. 172. Since decimals have the same scale of notation as whole numbers, they may be added, subtracted, multiplied, and divided in the same manner as whole numbers.

ADDITION OF DECIMALS.

Art. 173. To find the sum of two or more decimals.

Ex. 1. What is the sum of 37.58 + 2.9 + 45.084 + *312.0056 + .0248?* Ans. 397.5944.

WRITTEN PROCESS.
37.58
2.9
45.084
312.0056
.0248
Sum, 397.5944

EXPLANATION.

Since like quantities only can be added, the *tenths* are written in one column, the *hundredths* in another, &c., as in whole numbers. The sum of the right-hand column is 14 ten-thousandths, equal to 1 thousandth and 4 ten-thousandths. Write the 4 ten-thousandths in the place of that name, and carry the 1 thousandth to the column of that name, making its sum 14 thousandths, equal to 1 hundredth and 4 thousandths. Write the 4 thousandths in the place of that name, and carry the 1 hundredth to the column of that name, making its sum 19 hundredths, &c. The sum of the tenths is 15, equal to 1 *unit* and 5 *tenths*. Write the 5 tenths in the tenths' place, and carry the one unit to the column of units, &c. That is, the style of carrying is the same from the decimal to the integer as in any other part.

Rule.—*Write the numbers so that figures of the same denomination shall form a column. Add as in simple whole numbers, and put the decimal point in the result directly under those in the numbers added.*

PROOF.—As in Simple Addition.

EXAMPLES FOR PRACTICE.

2. Add .225, .415, .53, .314, .812, and .7. Ans. 2.996.
3. Add .314, .16723, .43417, .7351, and .307.
4. Add 2.531, 23.4327, 65.8162, and 13.5201. Ans. 105.3.
5. Add 15.3, 13.07, 9.407, 6.215, 5.013, and 10.
6. Add 41.6341, 15.15, .28173, 10.41003, and 24.
7. What is the sum of 15 and 43 hundredths, 437 thousandths, 25 and 704 thousandths, 30 and 7 tenths, and 506 thousandths?
8. What is the sum of 304 thousandths, 5103 hundred-millionths, 61032 millionths, 413 hundred-thousandths, and 603 ten-thousandths?
9. The distance on a railroad from A to B is 15.35 miles; from B to C is 24.268 miles; from C to D is 102.73 miles; and from D to E is 62.754 miles. What is the distance from A to E?

10. Jones bought 4 loads of hay, weighing 1.475 tons, 2.085 tons, 1.516 tons, and 1.435 tons, respectively; how many tons in all?

SUBTRACTION OF DECIMALS.

Art. 174. To subtract one decimal from another.

Ex. 1. From 15.625 take 7.9786 Ans. 7.6464.

WRITTEN PROCESS.	
	15.625
	7.9786
Rem.	7.6464

EXPLANATION.

Since a quantity can be taken only from a like quantity, *tenths* must be taken from *tenths*, *hundredths* from *hundredths*, *&c.* Hence, the subtrahend is placed under the minuend so that each figure is under a place of the same name. Subtraction and carrying proceed from right to left, as in whole numbers, and for the same reasons. Thus, 6 ten-thousandths from no ten-thousandths, impossible. Add 10: 6 from 10 leaves 4. Write 4, and carry 1 to 8, making 9: 9 from 5, impossible; 9 from 15 leaves 6. Write 6, and carry 1, &c.

Rule—*Write each figure of the subtrahend under the place of the same name in the minuend. Subtract as in simple whole numbers, and put the decimal point of the result under those of the given numbers.*

PROOF.—As in Simple Subtraction.

EXAMPLES FOR PRACTICE.

2. From 8.275 take 5.185. Ans. 3.09.
3. From 2.0132 take 1.25.
4. From 100 take .01. Ans. 99.99.
5. From 2000 take 20.0002. Ans. 1979.9998.
6. From 36.75 take 15.48.
7. From 3 take .214.
8. From 70.5 take .0375.
9. A man owes \$73. After paying \$7.3 how much will he still owe?

10. A man owning 875 acres of land, divided it among his four sons, as follows: to the first he gave 213.65 acres; to the second, 192.375 acres; to the third, 206.625 acres; and to the fourth, the remainder. What was the fourth son's share?

MULTIPLICATION OF DECIMALS.

Art. 175. To find the product of two decimals, or of a decimal and a whole number.

Ex. 1. Multiply .3 by .2. Ans. .06.

DEMONSTRATION.—Since $.3 = \frac{3}{10}$, and $.2 = \frac{2}{10}$, it is plain that the product of .3 by $.2 = \frac{3}{10} \times \frac{2}{10} = \frac{6}{100} = .06$.

Ex. 2. Multiply 2.05 by .003. Ans. .00615.

WRITTEN PROCESS.

$$\begin{array}{r} 2.05 \\ .003 \\ \hline \text{Prod. } .00615 \end{array}$$

DEMONSTRATION.

Since $2.05 = 2\frac{5}{100} = \frac{205}{100}$, and $.003 = \frac{3}{1000}$, it is plain that the product of 2.05 by $.003 = \frac{205}{100} \times \frac{3}{1000} = \frac{615}{100000} = .00615$.

Ex. 3. Multiply 4.375 by 6. Ans. 26.25.

DEMONSTRATION.—$4.375 \times 6 = \frac{4375}{1000} \times 6 = \frac{26250}{1000} = 26.25$.

Ex. 4. Multiply .0016 by 1000. Art. 1.6.

DEMONSTRATION.—$.0016 \times 1000 = \frac{16}{10000} \times 1000 = \frac{16000}{10000} = 1.6$.

From these examples it appears that, if the denominators of two decimals are expressed, their product has as many ciphers as there are in both denominators. Since there are as many places in a decimal as there are ciphers in its denominator, (See Art. 167), *the product of decimals has as many decimal places as there are in all the factors.*

Rule.—*Multiply as in simple whole numbers, and point off, if possible, from the right of the product, as many figures for decimals as there are decimal places in both factors. When there are not so many figures in the product, prefix ciphers, to supply the deficiency.*

PROOF.—As in Simple Multiplication.

NOTE 1.—After fixing the decimal point, erase the right-hand ciphers of the decimal part of the product, if there are any, as they do not affect its value.

NOTE 2.—To multiply by a power of 10, (See Art. 46) remove the decimal point of the multiplicand as many places toward the right as there are ciphers in the multiplier. If there are not enough figures in the number for this, annex ciphers to supply the deficiency.

EXAMPLES FOR PRACTICE.

	Multiply	Ans.		Multiply	Ans.
5.	.56 by .8.	.488.	15.	12.5 by 12.5.	156.25.
6.	.36 by .06.	.0216.	16.	.23 by 3.5.	
7.	3.003 by .002.	.006006.	17.	6.125 by 3.02.	
8.	2.375 by 8.	19.	18.	62.64 by .225.	
9.	23.75 by 8.		19.	.001 by .0001.	
10.	.05 by .05.	.0025.	20.	.0003 by .0002.	
11.	.0175 by 100.		21.	.0005 by 10.000.	
12.	43.75 by 1000.	43750.	22.	.0005 by 100.	
13.	.075 by .07.		23.	10000 by .0037.	
14.	12.25 by 12.		24.	4.024 by 100.	

25. At the rate of 2.375 miles per hour, how far will a man walk in 10 hours? Ans. 23.75

26. If a ship sail 31.25 miles per hour, how far will it sail in 24 hours?

27. What cost 3.25 tons of hay at $32.5 per ton?

28. What cost 16.27 acres of land at $50.6 per acre?

29. At $2.86 per ton, what will 7.5 tons of coal cost?

30. A man owning .6 of a mill, sold .25 of his share. What part of the mill did he sell? Ans. .15.

31. A man started on a journey of 325 miles. How far will he have to go after he has traveled 11 days at the rate of 25.23 miles per day?

32. A purchased goods as follows: 12.5 yards broadcloth at $4.75 per yard, 32 yards muslin at $.155 per yard, 2.3 yards velvet at $6 per yard, and 20.5 yards ticking at $.1875 *per yard.* What was the amount of his bill?

Art. 176. Since those figures of a decimal which are beyond the eighth or tenth place have a very small value, it is customary to neglect them in common results. For most purposes a decimal of four or five places in final results is deemed sufficiently accurate. If, therefore, the factors, or the divisors and dividends of a problem are decimals of many places, it is desirable to omit all computation that does not affect the required part of the result. For this purpose, methods of contracting the multiplication and division of decimals have been invented.

CONTRACTION IN MULTIPLICATION OF DECIMALS.

Art. 177. To multiply one decimal by another, so as to produce a required number of decimal places without unnecessary labor.

Ex. 1. Multiply 6.375179 by 8.264387, retaining five decimal places in the product. Ans. 52.68694.

Common Process.		First Short Process.		Second.	
	6.375179	6.37517	9	6.375179	
	8.264387	8.	264387	7834628	
4	4626253	5100143	2	5100143	2
51	001432	127503	5	127503	5
191	25537	38251	0	38251	0
2550	0716	2550	0	2550	0
38251	074	191	2	191	2
127503	58	51	0	51	0
5100143	2	4	4	4	4
52.68694	6 450273	52.68694		52.68694	

On inspecting the full common process, we perceive that the last right-hand figure required, namely 4, the fifth decimal figure from units, is the sum of the figures obtained by the multiplication of the 8 of the multiplier into 7 of the multiplicand, 2 of the multiplier into 1 of the multiplicand, 6 of the multiplier into 5 of the

multiplicand, 4 of the multiplier into 7 of the multiplicand, 3 of the multiplier into 3 of the multiplicand, and 8 of the multiplier into 6 of the multiplicand, *plus, in each case, respectively, what must be brought in by carrying from the previous multiplication.* Therefore we can neglect such parts of the process as produced the figures which are on the right of the vertical line. To do this we can proceed as shown in the short processes, and explained in the following

Rules.—I. *Draw a vertical line so that as many decimal places of the multiplicand are on the left of the line as there are decimal places required in the product. If there are not so many decimal places in the multiplicand, supply the deficiency with ciphers.*

Write the multiplier under the multiplicand, so that its place of units is next left of the line.

Beginning at the left of the multiplier, form the partial products by starting with that figure of the multiplicand which is as many places from the line on one side as the figure by which you multiply is on the other side. If there is no such starting figure, supply with ciphers.

After adding in the figure that must be carried from the product of the figure in the next lower place of the multiplicand, write the first figure of each partial product on the right of the line, and the other figures in order on the left.

Add the partial products; reject the units in the sum of the column on the right of the line, but carry its tens, and point off the required number of places from the figures on its left.

II. *Write the multiplier so that its units' place is under that place of the multiplicand, which is the same as the last required place of the product, and write the other figures of the multiplier in reverse order toward the left.*

Multiply each figure of the multiplier into that part of the multiplicand which begins one place to the right of that figure, putting in one column the first figure of each partial product, after it has been increased by the figure carried from the product of the figure in the next lower place.

Add the partial products, rejecting the sum of the right-hand column, but carrying its tens; and from the right of the whole sum point off the required number of places of decimals.

NOTE.—It is easily perceived that both processes are identical in the computation, differing only in the way of arranging the multiplier.

EXAMPLES FOR PRACTICE.

2. Multiply 9.5002 by .005732, retaining 5 decimal places in the product. Ans. .05445.

3. Multiply 3.012 by 16.25, retaining 1 decimal place in the product. Ans. 48.8.

4. Multiply 23.13254 by 6.4123, retaining 6 decimal places in the product.

5. Multiply 268.19222 by 5.69377, retaining 4 decimal places in the product. Ans. 1527.0257.

6. Multiply 4.237 by .61347, retaining 3 decimal places in the product.

7. Multiply 424.96875 by .084888, retaining 3 decimal places in the product. Ans. 36.075.

8. Multiply .721433 by .01124, retaining 5 decimal places in the product.

9. Multiply 2.64814 by 11.07, retaining 4 decimal places in the product. Ans. 29.3150.

10. Multiply 7.1325 by 6.1345, retaining 2 decimal places in the product.

DIVISION OF DECIMALS.

Art. 178. To divide a decimal by a decimal, a decimal by an integer, or an integer by a decimal.

The dividend is equal to the product of the divisor and quotient. But a product has as many decimal places as there are in the factors. (See Art. 175). Therefore the dividend has as many decimal places as there are in both divisor and quotient. Therefore the quotient has as many decimal places as those in the dividend exceed those in the divisor.

Ex. 1. Divide .0072 by .06. **Ans.** .12.

WRITTEN PROCESS. DEMONSTRATION.

$$.06)\,.0072 \quad \overline{\;.12\;}$$

$$\tfrac{72}{10000} \div \tfrac{6}{100} = \tfrac{72}{10000} \times \tfrac{100}{6} = \tfrac{12}{100} = .12.$$

Ex. 2. Divide .0072 by 6. **Ans.** .0012.

WRITTEN PROCESS. DEMONSTRATION.

$$6)\,.0072 \quad \overline{\;.0012\;}$$

$$\tfrac{72}{10000} \div 6 = \tfrac{12}{10000} = .0012.$$

Ex. 3. Divide 72 by .06. **Ans.** 1200.

WRITTEN PROCESS. DEMONSTRATION.

$$.06)\,72.00 \quad \overline{\;1200\;}$$

$$72 \div \tfrac{6}{100} = 72 \times \tfrac{100}{6} = \tfrac{7200}{6} = 1200.$$

Ex. 4. Divide .0072 by 100. **Ans.** .000072.

DEMONSTRATION.

$$.0072 \div 100 = \tfrac{72}{10000} \div 100 = \tfrac{72}{1000000} = .000072.$$

Rule.—*Divide as in simple whole numbers, and point off, if possible, as many figures from the right of the quotient, for decimals, as the decimal places of the dividend exceed those of the divisor. When there are not so many figures in the quotient, prefix ciphers to supply the deficiency.*

PROOF.—As in Simple Division.

NOTE 1.—When there are not so many decimal places in the dividend as in the divisor, make them as many by annexing ciphers. The quotient is then a whole number.

NOTE 2.—When there is a remainder, annex ciphers as decimals to the dividend, and continue the division. If a perfect quotient is not thus obtained, annex the sign + to it, to signify that the quotient is not complete.

NOTE 3.—To divide by a power of 10, (See Art. 64) remove the decimal point of the dividend as many places towards the left as there are ciphers in the divisor. If there are not enough figures in the number for this, prefix ciphers to supply the deficiency. (See Ex. 4).

NOTE 4.—Experience has shown that the proper management of the decimal point in multiplication and division is one of the most difficult to secure in learners. Hence, much pains should be taken in this matter, and close, varied, and careful drill maintained till skill and accuracy are attained.

EXAMPLES FOR PRACTICE.

Divide

5. 62.5 by 2.5. Ans. 25.
6. 1.728 by .12. Ans. 14.4.
7. 25 by .05. Ans. 500.
8. 364 by .004.
9. 126 by .0003.
10. 72 by .24.
11. .72 by 24.
12. 104 by .026.
13. 108 by 1.8.
14. 365 by .73.
15. 40 by 16. Ans. 2.5.
16. 16 by 40. Ans. .4.
17. 31.25 by 25.
18. .064 by 8.

Divide

19. 25 by 500. Ans. .05.
20. .875 by .35.
21. .15 by 500.
22. 100 by .01.
23. .01 by 100.
24. 10 by .0001.
25. 2 by .05.
26. 6 by .4.
27. 20.6 by 16.
28. 3241 by 6.4.
29. 3.241 by 100000.
30. .1 by .064.
31. .071 by .08.
32. 31.5 by 3.2.

33. A man divided 46.5 acres of land into 3 equal lots: how many acres did each lot contain?

34. If 16 men mow 28.4 acres of grass in a day, how much does each man mow?

35. At $.25 per pound, how many pounds of butter can be bought for $3?

36. If a man earns $113.75 in 7 weeks, how much is that per week?

37. A man has 324 bushels of apples, which he wishes to put into barrels, containing 2.25 bushels each. How many barrels will be required?

38. If 8 men dig a ditch 810.6 feet long in 7.5 days, how much will 1 man dig in 1 day? Ans. 13.51.

39. There are 16.5 feet in one rod, and 5280 feet in a mile; how many rods in a mile?

40. How many bushels of clover-seed at $6.25, will pay for 25 barrels flour at $10.5 per barrel?

41. Multiply the sum of 18.75 and 24.3 by their difference and divide the product by 1.5625.

CONTRACTION IN DIVISION OF DECIMALS.

Art. 179. To divide one decimal by another, so as to produce a required number of decimal places without unnecessary labor?

Ex. 1. Divide 25.631549287 by 4.30527, finding the quotient to three decimal places.

COMMON PROCESS.

```
4.30527)25.631|549287(5.953
        21526 |35
        ------|------
         4105 |199
         3874 |743
         -----|------
          230 |4562
          215 |2635
          ----|------
           15 |19278
           12 |91581
           ---|------
            2 |27697
```

SHORT PROCESS.

```
4.305)25.631(5.953
      21526
      -----
       4105
       3874
       ----
        231
        216
        ---
         15
         13
         --
          2
```

EXPLANATION.

On inspecting the full common process, we perceive that those parts of divisor and dividend which are beyond the third decimal place do not affect the quotient, except by what is carried to what is immediately under the third decimal place. Therefore we can neglect those parts in dividing, except so far as reckoning their carrying figures. This is done in the Short Process here given, and a comparison of it with the full process will be enough to show the reason of every point in the following rule. By inspection we perceive that the first figure, 5. of the quotient must be *units*, because the integral part 4, of the divisor, is contained in the integral part 25, of the dividend, less than ten times.

Rule.—*First determine how many integral figures there will be in the quotient, then take for a divisor as many of the left-*

hand figures of the divisor as equal the number of quotient figures, including the required number of decimal places of the quotient. If there are not so many figures in the divisor, annex decimal ciphers to supply the deficiency.

Annex no new figure to any remainder, but, at each new division, use the preceding divisor, except its right-hand figure, and in forming the products add in what would have been carried from the product of the rejected figures.

NOTE.—The number of integral figures in the quotient is determined by finding the *order* of its first figure. This can be done by making a trial with the integral parts of divisor and dividend. For example, by varying the place of the decimal point in example 1, we would obtain results like the following:—

4.30527) 2.5631549287 (.5953 +
4.30527) .25631549287 (.0595 +
4.30527) 25.631549287 (5.953 +
4.30527) 256.31549287 (59.535 +
4.30527) 2563.1549287 (595.352 +
.430527) 25.631549287 (59.535 +
.0430527) 25.631549287 (595.352 +

In these it is easy to see the local value, or order, of the first figure of the quotient, when we try 4 in 2, 4 in .2, 4 in 25, 4 in 256, 4 in 2563, .4 in 25, .04 in 25, &c. In each case, as the required number of decimal places in the quotient is three, we use, by the rule, three more places of the divisor than there are integral figures in the quotient.

EXAMPLES FOR PRACTICE.

2. Divide 9.12606 by 1.44223, finding the quotient to 3 decimal places. Ans. 6.328.

3. Divide 7.64352 by .181543, finding the quotient to 2 decimal places.

4. Divide 200 by 1.8660254, finding the quotient to 5 decimal places. Ans. 107.17968.

5. Divide 31.2372 by 8.15325, finding the quotient to 4 decimal places.

6. Divide 16.3415 by 24.6321, finding the quotient to 3 decimal places.

REDUCTION OF DECIMALS.

Art. 180. Reduction of decimals is changing their form without altering their value. It also includes the change of common fraction to the form of a decimal.

A decimal is said to be reduced to *higher decimal terms* when its written portion, or numerator, is made a greater number. Its last place then expresses a lower order of value. Thus, .5, or five *tenths* is, by annexing a cipher, made .50, a larger number of figures, but expressing *hundredths*, a lower order of value.

A decimal is said to be reduced to *lower terms* when its written portion, or numerator, is made a smaller number. Its last place then expresses a higher order of value.

Art. 181. Since ciphers on the right of a decimal do not change the places of its significant figures, and do not themselves express any value, *ciphers on the right of a decimal do not affect its value.*

Since every cipher placed between a decimal and the decimal point removes the decimal one place further from that point, *every cipher placed between a decimal and its decimal point causes it to express one-tenth of its former value.* Thus, .05 is one-tenth of .5.

Case I.

Art. 182. To reduce a decimal to another order.

Rules I.—To reduce to a lower order, *annex as many ciphers as will make the right-hand cipher occupy the place of that order.*

II. To reduce to a higher order, when the right-hand figures of a decimal are ciphers, *erase ciphers enough from the right of the decimal to make its right-hand figure occupy the place of the required order.*

Note.—If the right-hand figures are not ciphers, the second rule would make a *double decimal*, which is not allowable. Thus, to reduce .425 to *tenths*, if we cut off 25 by another point, we have .4.25, $= 4\frac{25}{100}$ *tenths, or* $4\frac{1}{4}$ *tenths*, a disadvantageous and unusual form of fraction.

Examples for Practice.

1. Reduce .65 to thousandths; to hundred-thousandths.
2. Reduce .7000 to thousandths; hundredths; tenths.
3. Reduce .6 to hundredths; ten-thousandths; millionths.
4. Reduce .29 to thousandths; ten-millionths.
5. Reduce .500000 to hundred-thousandths; hundredths.
6. Reduce .4300000 to millionths; ten-thousandths.
7. Reduce .8 to hundredths; thousandths; ten-thousandths.
8. Reduce .60000 to thousandths; hundredths; tenths.
9. Reduce .435 to millionths; hundred-millionths.

Case II.

Art. 183. To reduce two or more decimals to the same order, that is, to a common denominator.

Rule.—*Annex ciphers so that the decimals shall all have the same number of decimal places.*

Note.—When all the rest of a given set of decimals are thus made to have as many places as that which has the most, the set will have their least common decimal denominator.

Examples for Practice.

1. Reduce .75, .324, .0237, and .61586 to their least common decimal denominator.

2. Reduce .073, .24, .6275, and .00413 to their least common decimal denominator.

3. Reduce 2.4, 24.024, 62.5. 100.01 and .00006 to their least common decimal denominator.

4. Reduce .143, 1.43, 14.3, 143, and 1430 to their least common decimal denominator.

5. Reduce .75, .00162, .5, 500, and 87.5 to their least common decimal denominator.

CASE III.

Art. 184. To reduce a mixed decimal to the lowest order expressed in it.

Ex. 1. Reduce 5.187 to thousandths.

Ans. 5187 thousandths.

ANALYSIS.—Since 1 is equal to 1000 thousandths, 5 is equal to 5 times 1000 thousandths, that is, 5000 thousandths, and 187 thousandths more make the whole quantity 5187 thousandths.

Rule.—*Consider the entire expression as a whole number, with the name of its lowest order.*

EXAMPLES FOR PRACTICE.

2. Reduce 6.25 to hundredths. Ans. 625 hundredths.
3. Reduce 2.375 to thousandths. Ans. 2375 thousandths.
4. Reduce 5.6 to tenths.
5. Reduce 43.8045 to ten-thousandths.
6. Reduce 181.175 to thousandths.
7. Reduce 25.000025 to millionths.
8. Reduce 14.00014 to hundred-thousandths.

CASE IV.

Art. 185. To reduce a whole number to a decimal of a given order.

Ex. 1. Reduce 4 to tenths. Ans. 40 tenths, or 4.0.

ANALYSIS.—Since 1 is equal to 10 tenths, 4 is equal to 4 times 10 tenths, that is, 40 tenths, or 4.0.

Rule.—*Annex as many decimal ciphers as the given decimal has places.*

EXAMPLES FOR PRACTICE.

2. Reduce 5 to tenths; hundredths; thousandths.
3. Reduce 27 to ten-thousandths; hundred-thousandths.
4. Reduce 12 to thousandths; millionths.

CASE V.

Art. 186. To reduce a decimal to a common fraction.

Ex. 1. Reduce .0625 to a common fraction. Ans. $\frac{1}{16}$.

PROCESS INDICATED.—$.0625 = \frac{625}{10000} = \frac{5}{80} = \frac{1}{16}$.

Rule.—*Erase the decimal point, and express the decimal as a whole number; then write the denominator under it, and reduce the fraction thus made to its lowest terms.*

EXAMPLES FOR PRACTICE.

Reduce the following decimals to common fractions.

2. .375. Ans. $\frac{3}{8}$.
3. .64. Ans. $\frac{16}{25}$.
4. .016.
5. .225.
6. .0875.
7. .1375.
8. .7125.
9. $.2\frac{1}{2}$. Ans. $\frac{1}{4}$.
10. $.16\frac{2}{3}$. Ans. $\frac{1}{6}$.
11. $.37\frac{1}{2}$.
12. 3.75. Ans. $3\frac{3}{4}$.
13. 16.325.
14. $12.2\frac{1}{4}$. Ans. $12\frac{9}{40}$.
15. $10.62\frac{1}{2}$.
16. 18.75.
17. $3.43\frac{3}{4}$.
18. What part of \$1 is \$.65? Ans. $\frac{13}{20}$.
19. What part of a mile is $.68\frac{3}{4}$ of a mile? Ans. $\frac{11}{16}$.
20. What part of a ton is $.15\frac{5}{8}$ of a ton?
21. What fractional part of a mile $= .66\frac{2}{3}$ of a mile?
22. A man sold $.62\frac{1}{2}$ of his farm. What fractional part is that?

CASE VI.

Art. 187. To estimate approximately the value of a decimal of many places.

Ex. 1. What is nearly the value of .39901? Ans. $\frac{2}{5}$.

EXPLANATION.—Since .399 differs only $\frac{1}{1000}$ from .4, and since .00001 expresses very little additional value, the given decimal does not differ much from $.4 = \frac{4}{10} = \frac{2}{5}$.

Rule.—*Reduce to a common fraction the first two or three figures.*

EXAMPLES FOR PRACTICE.

2. What is nearly the value of .24934? Ans. $\frac{1}{4}$.
3. What is the approximate value of .45132?
4. What is the approximate value of .74891?
5. What is nearly the value of .39987?
6. What is nearly the value of .65024?
7. What is nearly the value of .3750316? Ans. $\frac{3}{8}$.
8. What is nearly the value of .8749302?

CASE VII.

Art. 188. To reduce a common fraction to a decimal.

Ex. 1. Reduce $\frac{1}{16}$ to a decimal. Ans. .0625.

WRITTEN PROCESS.	EXPLANATION.
.16(1.0000 .0625	One-sixteenth of 1 equals one-sixteenth of 10000 ten-thousandths, which is 625 ten-thousandths, or .0625.

Rule.—*Annex ciphers as decimals to the numerator, and divide this result by the denominator.*

NOTE.—Every new decimal figure thus obtained makes a nearer approximation to the value of the common fraction. When enough decimal figures have been obtained for our purpose, we can annex + if there is a remainder. Some prefer to annex + if the next decimal figure would be less than 5, but, if it would be more than 5, to increase the last decimal figure by 1, and annex —.

EXAMPLES FOR PRACTICE.

2. Reduce $\frac{5}{8}$ to a decimal. Ans. .625.
3. Reduce $\frac{3}{4}$ to a decimal.
4. Reduce $\frac{9}{16}$ to a decimal. Ans. .5625.
5. Reduce $\frac{13}{25}$ to a decimal. Ans. .52
6. Reduce $\frac{17}{32}$ to a decimal.
7. Reduce $\frac{31}{640}$ to a decimal.

8. Reduce $\frac{11}{24}$ to a decimal. **Ans.** .4583 +.
9. Reduce $\frac{137}{625}$ to a decimal.
10. Reduce $\frac{41}{125}$ to a decimal.
11. Reduce $\frac{75}{1280}$ to a decimal.

REPEATING, CIRCULATING, OR PERIODICAL DECIMALS.

Art. 189. Those decimals which result from the division of one number by another are either *finite* or *infinite.*

A **finite,** or **terminate,** decimal is a decimal which terminates. Thus, $\frac{1}{4}$ produces .25, a finite decimal.

An **infinite,** or **interminate,** decimal is a decimal which does not terminate. Thus, $\frac{1}{3}$ produces .333 +, which can never be completed by prolonging the division.

Art. 190. Infinite decimals are either *repeating,* or *non-repeating.*

A **repeating decimal** is a decimal whose figures are repetitions of the same figure, or set of figures. Thus, .6666 + is a repeating decimal whose figures are repetitions of 6, and the decimal .272727 + is repeating, its figures being repetitions of 27.

Repeating decimals are also called **recurring** decimals, because the same figure, or set of figures, constantly recurs. They are also called **periodical decimals,** because their figures are arranged in definite collections, called *periods;* and **circulating decimals,** because their development is like motion in a circle, which constantly returns to the same place.

A **non-repeating** decimal is a decimal whose figures are not repetitions of the same figure, or set of figures. The ratio of the diameter to the circumference of a circle is a specimen of an infinite non-repeating decimal.

A **repetend** is the figure, or set of figures, whose repetitions compose a repeating decimal. Thus, 6 and 27 are the repetends in .6666 + and .272727 +.

Art. 191. A repeating decimal is denoted by a *dot* over the figure of the repetend, if it is single, or over the first and last figures, if there are two or more. Thus, .6666 + may be denoted by $.\dot{6}$; .272727 + may be denoted by $.\dot{2}\dot{7}$; and .270270 + may be denoted by $.\dot{2}7\dot{0}$.

Art. 192. A **pure repeating decimal** is a decimal entirely composed of a repetend. Thus, $.\dot{7}$ and $.\dot{2}7\dot{9}$ are pure repeating decimals, being equal to .7777 + and .279279 +.

A **mixed repeating decimal** is a decimal whose first portion is not a repetend, but is followed by a repetend. Thus, $.21\dot{3}$ is a mixed repeating decimal, being equal to .213333 +.

Art. 193. **Similar repetends** are repetends which begin at the same decimal place. Thus, $.02\dot{2}\dot{7}$ and $.16\dot{7}6190\dot{4}$ are similar repetends, because both begin at the place of thousandths.

Dissimilar repetends are repetends which begin at different decimal places. Thus, $.8\dot{2}7\dot{9}$ and $.37\dot{4}2\dot{3}$ are dissimilar repetends, because one begins at the place of hundredths, and the other at that of thousandths.

Conterminous repetends are repetends which end at the same decimal place. Thus, $.85\dot{4}\dot{2}$ and $.0\dot{1}2\dot{3}$ are conterminous, because both repetends end at the fourth decimal place.

Similar and conterminous repetends are repetends which begin at the same decimal place, and end at the same decimal place.

REDUCTION OF REPEATING DECIMALS.

CASE I.

Art. 194. To determine the kind of decimal arising from a given fraction.

Proposition I.—*Finite decimals arise from those common fractions which, in their lowest terms, have no other prime factors in the denominator than 2 and 5.*

DEMONSTRATION.—Since a common fraction is reduced to a decimal by annexing ciphers to the numerator, that is, by multiplying its numerator by a power of 10, whose component factors are 2 and 5, and dividing by the denominator, it is plain that a denominator which has no other prime factors than 2 or 5 will be exactly contained in some extension of the numerator by ciphers.

ILLUSTRATION.—Three-eighths $= \frac{3}{2 \times 2 \times 2}$. To convert this into a decimal, we multiply the numerator, 3, by $1000 = 10 \times 10 \times 10 = 2 \times 5 \times 2 \times 5 \times 2 \times 5$, making the decimal equivalent to

$$\frac{3 \times \cancel{2} \times 5 \times \cancel{2} \times 5 \times \cancel{2} \times 5}{\cancel{2} \times \cancel{2} \times \cancel{2} \times 1000} = \frac{3 \times 5 \times 5 \times 5}{1000} = \frac{375}{1000} = .375.$$

Again, $\frac{3}{25} = \frac{3}{5 \times 5}$. To convert this into a decimal, we multiply the numerator, 3, by $100, = 10 \times 10 = 2 \times 5 \times 2 \times 5$, making the decimal equivalent to

$$\frac{3 \times 2 \times \cancel{5} \times 2 \times \cancel{5}}{\cancel{5} \times \cancel{5} \times 100} = \frac{3 \times 2 \times 2}{100} = \frac{12}{100} = .12.$$

COROLLARY.—*Infinite decimals arise from those common fractions which, in their lowest terms, have in the denominator other prime factors than 2 and 5.*

DEMONSTRATION.—Because the prime factors of the denominator are not 2 or 5, they cannot be exactly contained in any number of times 2 times 5 times the numerator, that is, in the extension of the numerator by ciphers.

Proposition II.—*Every infinite decimal which arises from a common fraction is a repeating decimal.*

DEMONSTRATION.—In dividing the numerator of a common fraction by its denominator, any remainder must be less than the denominator. Hence, all the possible remainders must be among the numbers from 1 to 1 less than the denominator. Hence, if the division be carried far enough, a figure of the remainder must *recur*, and, if it recurs, the other remainders must recur in order, and, consequently, the quotient figures must recur in order.

COROLLARY.—The number of places in a repetend must be less than the number of units in the denominator of its generating fraction.

Rule.—*Reduce the given fraction to its lowest terms, and resolve the denominator into its prime factors. If these factors are 2, or 5, or both, the decimal will be finite. If other prime factors occur with 2, or 5, or both, the decimal will be a mixed repeating decimal. If none of the prime factors are 2 or 5, the decimal will be a pure repeating decimal.*

In the case of a mixed repeating decimal, the number of figures preceding the repetend will equal the greatest number of times that one of the factors 2 or 5 occurs in the denominator.

The number of figures in the repetend equals the number of times that a factor of the denominator, other than 2 or 5, must be used in dividing 1 with ciphers annexed, before the remainder 1 occurs.

Illustrations.

1. The fraction $\frac{13}{70}$, whose denominator equals $2 \times 5 \times 7$, produces the mixed repetend $.1\dot{8}5714\dot{2}$, there being but one figure preceding the repetend, and one occurrence of 2 or 5 in the denominator. Also, the repetend has six figures, which is the number of times that the factor 7 of the denominator must be used in dividing 1 with ciphers annexed, before the remainder 1 occurs.

2. The fraction $\frac{67}{84}$, whose denominator equals $2 \times 2 \times 3 \times 7$, produces the mixed repetend $.79\dot{7}6190\dot{4}$, there being two figures preceding the repetend, and two occurrences of two in the denominator. Also, there are six figures in the repetend, for the reason given in the first illustration.

Examples for Practice.

Reduce the following fractions to decimals, tell the kind of decimal produced, and mark the repetend if repeating.

1. $\frac{5}{8}$.	7. $\frac{11}{63}$.	13. $\frac{35}{64}$.
2. $\frac{5}{9}$.	8. $\frac{12}{35}$.	14. $\frac{21}{27}$.
3. $\frac{5}{11}$. Ans. $.\dot{4}\dot{5}$.	9. $\frac{41}{49}$.	15. $\frac{93}{250}$.
4. $\frac{7}{12}$. Ans. $.58\dot{3}$.	10. $\frac{17}{25}$.	16. $\frac{23}{27}$.
5. $\frac{18}{21}$.	11. $\frac{2}{13}$.	17. $\frac{13}{80}$.
6. $\frac{19}{32}$.	12. $\frac{4}{185}$.	18. $\frac{4}{17}$.

Case II.

Art. 195. To reduce a repeating decimal to an equivalent common fraction.

Proposition.—*Every repetend is equivalent to a fraction whose numerator is the repetend, and whose denominator is as many 9's as the repetend has figures.*

Demonstration.—Ten times a pure repetend of one figure, for example .3333 +, is a whole number and a repetend of the same figure, thus, 3.333 +. Therefore the whole part is 9 times the repetend. Therefore the value of the repetend is $\frac{1}{9}$ of that figure. In the example, 3 is 9 times .333 +, and therefore .333 + is equal to $\frac{3}{9}$, or $\frac{1}{3}$. In the same manner it may be proved that 100 times a pure repetend of two figures has an integral part which is the same figures, and which is 99 times the value of the repetend, and, hence, that the repetend is equal to those figures over 99; &c.

Ex. 1. What is the value of $.70\dot{4}\dot{5}$? Ans. $\frac{31}{44}$.

Analysis.—The non-repeating part .70, is equal to $\frac{70}{100}$, and the repetend, 45, is equal to $\frac{45}{99}$. Therefore the whole decimal is equal to $70\frac{45}{99}$ *hundredths*, or the common fraction $\frac{6975}{9900}$, or $\frac{31}{44}$.

Ex. 2. What is the value of $\dot{1}2.\dot{3}$? Ans. $12\frac{104}{333}$.

Analysis.—Expanding the expression, $\dot{1}2.\dot{3} = 12.3\dot{1}2\dot{3}$. The integral part, 12, and the .3 equal $12\frac{3}{10}$; the repetend, 123, is equal to $\frac{123}{999}$ *tenths*. Therefore the whole expression is equal to 12, + $3\frac{123}{999}$ *tenths*, or $12\frac{3120}{9990}$, or $12\frac{312}{999}$, or $12\frac{104}{333}$.

Note.—This result would have been reached more speedily by considering the whole number as 12, and the repetend as a decimal beginning at tenths, viz.: $.\dot{3}1\dot{2}$, making $12\frac{312}{999} = 12\frac{104}{333}$.

Rule.—To find the value of a pure repeating decimal, *write the repetend for a numerator, and for the denominator as many 9's as the repetend has figures.*

To find the value of a mixed repeating decimal, *write its finite part as an integer, its repetend as a fraction with the proper number of 9's, and both as a mixed number for the numerator of a common fraction, whose denominator is that power*

of 10 which expresses the place of the last finite figure, then reduce this complex fraction to a simple fraction.

NOTE.—Hence, in practice, *for a numerator subtract the finite part from the mixed expression; make the denominator as many 9's as the repetend has figures, with as many ciphers as there are figures in the finite part.*

EXAMPLES FOR PRACTICE.

3. Reduce $.\dot{4}\dot{5}$ to a common fraction. Ans. $\frac{5}{11}$.
4. Reduce $.\dot{2}\dot{7}$ to a common fraction.
5. Reduce $.\dot{2}9\dot{7}$ to a common fraction.
6. Reduce $.\dot{2}8571\dot{4}$ to a common fraction. Ans. $\frac{2}{7}$.
7. What is the value of $5.\dot{6}\dot{3}$. Ans. $5\frac{7}{11}$.
8. What is the value of $15.\dot{3}8461\dot{5}$.
9. Reduce $.3\dot{8}\dot{1}$ to a common fraction. Ans. $\frac{21}{55}$.
10. Reduce $.2\dot{6}\dot{9}$ to a common fraction.
11. Reduce $.10\dot{3}\dot{6}$ to a common fraction.

CASE III.

Art. 196. To reduce dissimilar to similar repetends.

Proposition.—*The value of an infinite repeating decimal is not affected by changing the place of beginning of the repetend.*

DEMONSTRATION.—Since the value of the figures of a decimal depends on their place, as reckoned from units, and the changing of their *order* in a repetend does not change their *place*, the shifting of the place of beginning of a repetend does not affect the value of the decimal.

COROLLARY 1.—Any pure repeating decimal may be made a mixed one, by beginning the repetend further to the right. Thus, $.\dot{2}\dot{3} = .2\dot{3}\dot{2}$, or $.23\dot{2}\dot{3}$, or $.232\dot{3}\dot{2}$, &c.

COROLLARY 2.—Small repetends may be united into a larger. Thus, $.\dot{4}\dot{6}$ may be $.\dot{4}64\dot{6}$, or $.\dot{4}6464\dot{6}$, &c.

Rule.—*Change the dots of the repetends so that they may begin at the same decimal place, taking care that one repetend at least in each decimal shall be fully expressed.*

EXAMPLES FOR PRACTICE.

1. Reduce $.\dot{1}4285\dot{7}$, $.53\dot{2}\dot{7}$, $.25\dot{6}$, $.4\dot{2}7\dot{9}$, and $.231\dot{6}\dot{3}$ to similar repetends.

2. Reduce $.7\dot{3}$, $.43\dot{3}\dot{6}$, $.133\dot{4}\dot{5}$, $.203\dot{6}4\dot{8}$, and $.\dot{8}\dot{1}$ to similar repetends.

3. Reduce $.3127\dot{8}3\dot{7}$, $.632\dot{7}0\dot{9}$, $.\dot{4}$, $.23\dot{5}\dot{4}$, and $.403\dot{1}29\dot{6}$ to similar repetends.

4. Reduce $\frac{11}{15}$, $\frac{13}{18}$, $\frac{9}{14}$, $\frac{5}{8}$, and $\frac{5}{7}$ to decimals and make the repetends similar.

5. Reduce $\frac{2}{3}$, $\frac{3}{7}$, $\frac{7}{9}$, and $\frac{3}{4}$ to decimals and make the repetends similar.

CASE IV.

Art. 197. To make repeating decimals conterminous.

Ex. 1. Make $.\dot{2}\dot{3}$, $.4\dot{5}6\dot{7}$, and $.28\dot{7}58\dot{4}$ conterminous.

PROCESS INDICATED.

$$.\dot{2}\dot{3} = .2\dot{3}232323232\dot{3}$$
$$.4\dot{5}6\dot{7} = .45\dot{6}7567567567\dot{5}$$
$$.28\dot{7}58\dot{4} = .28\dot{7}5847584758\dot{4}$$

EXPLANATION.

These decimals are not similar, and they must first be made similar by Case III. To make them conterminous, we observe that the first repetend has *two* places, the second *three*, and the third *four*. Hence, if they *begin* at the same place, the first place at which their last figures will come together will be the least common multiple of 2, 3, and 4, that is, the twelfth place.

Rule.—*First make the given repetends similar, then extend them to the right till each contains as many places as there are units in the least common multiple of the number of places in the several repetends, and put a dot over the last figure thus found.*

EXAMPLES FOR PRACTICE.

2. Make $.\dot{6}$, $.\dot{7}\dot{2}$, $.\dot{8}4\dot{6}$, and $.53\dot{4}3\dot{2}$ conterminous.

3. Make $53.\dot{3}$, $.16\dot{5}\dot{4}$, $8.28\dot{7}\dot{5}$, and $.\dot{7}$ conterminous.

4. Make $85.\dot{8}$, $6.\dot{8}\dot{4}$, $.\dot{1}1\dot{7}$, and $.\dot{7}58\dot{7}$ similar and conterminous.

5. Make $.5046\dot{6}3\dot{3}$, $.\dot{4}\dot{2}$, $.4\dot{2}$, $.\dot{0}4\dot{8}$, and $.\dot{0}07\dot{5}$ similar and conterminous.

6. Reduce $\frac{4}{7}$, $\frac{3}{11}$, and $\frac{8}{9}$ to decimals, and make their repetends similar and conterminous.

ADDITION OF REPEATING DECIMALS.

Art. 198. To find the sum of two or more repeating decimals.

Ex. 1. What is the sum of $1.\dot{7}$, $4.7\dot{8}\dot{7}$, $2.03\dot{6}3\dot{1}$, and $6.1\dot{7}2\dot{8}$?

Ans. $14.77\dot{4}8457\dot{5}$.

WRITTEN PROCESS.

DISSIMILAR.	SIMILAR AND CONTERMINOUS.
$1.\dot{7}$	$= 1.77\dot{7}77777\dot{7}$
$4.7\dot{8}\dot{7}$	$= 4.78\dot{7}87878\dot{8}$
$2.03\dot{6}3\dot{1}$	$= 2.03\dot{6}3163\dot{1}$
$6.1\dot{7}2\dot{8}$	$= 6.17\dot{2}87287\dot{7}$
	$14.77\dot{4}8457\dot{5}$

EXPLANATION.

Since fractions can be added only when they express like parts of a unit, the repetends must first be made *similar*, because they then have a common denominator, which, if expressed, would be 99999900. They can then be added as common decimals. To the sum of the right-hand column must be carried as much as would be carried if the decimals were extended to form more columns. In this case 2 must be carried, making 25, of which we write the 5, and proceed as usual.

The result can be seen to be correct by finding the sum of the actual repetends, which is 2484573, and dividing by their common denominator, 99999900, giving $.02\frac{484575}{999999}$, which, added to 14.75, gives $14.77\frac{484575}{999999}$, $= 14.77\dot{4}8457\dot{5}$.

Rule.—*Make the repetends similar and conterminous, and add as in finite decimals, observing to increase the sum of the right-hand column by as many units as are carried from the left-hand column of the repetends. Make a repetend in the result similar and conterminous with those added.*

EXAMPLES FOR PRACTICE.

2. What is the sum of $.\dot{3}$, $.\dot{8}\dot{7}$, $.5\dot{6}4\dot{3}$, and $.\dot{1}\dot{2}$?
Ans. $1.8\dot{9}7\dot{6}$.
3. What is the sum of $.\dot{1}\dot{6}$, $.\dot{7}9\dot{2}$, $.2\dot{1}43\dot{1}$, and $.\dot{5}\dot{6}$?
4. What is the sum of $.\dot{0}\dot{9}$, $.20\dot{4}\dot{5}$, and $.25$? Ans. $.\dot{5}\dot{4}$.
5. Add $7.\dot{1}2494\dot{3}$, $5.0\dot{7}7\dot{0}$, and $.\dot{2}\dot{4}$.
6. Add $.\dot{6}\dot{1}$, $.\dot{2}\dot{5}$, $.56\dot{3}\dot{5}$, and $.10\dot{4}$. Ans. $1.53\dot{6}$.
7. Add $.\dot{4}\dot{3}$, $.\dot{3}72\dot{1}$, $8.5\dot{8}32\dot{6}$, and $7.\dot{5}$.
8. Add $.5\dot{3}\dot{2}$, $.10\dot{5}$, $.\dot{2}3\dot{1}$, $.204\dot{1}2\dot{3}$.

SUBTRACTION OF REPEATING DECIMALS.

Art. 199. To find the difference of two repeating decimals.

Ex. 1. From $16.\dot{7}\dot{2}$ take $5.41\dot{9}3\dot{7}$. Ans. $11.30\dot{7}8933\dot{4}$.

WRITTEN PROCESS.

DISSIMILAR.		SIMILAR AND CONTERMINOUS.
$16.\dot{7}\dot{2}$	$=$	$16.72\dot{7}27272\dot{2}$
$5.41\dot{9}3\dot{7}$	$=$	$5.41\dot{9}37937\dot{7}$
		$11.30\dot{7}8933\dot{4}$

EXPLANATION.

Since a fraction can only be taken from another which expresses like parts of a unit, the given repetends must first be made similar and conterminous, because they then have a common denominator, which, if expressed, would be 99999900. Then we can subtract as in common decimals, observing to carry to the first right-hand figure, 7, the 1 which would come from the extended decimals. The result is seen to be correct by taking $5.41\frac{937937}{999999}$ from $16.72\frac{727272}{999999}$, giving $11.30\frac{789334}{999999}$.

Rule.—*Make the repetends similar and conterminous, and subtract as in finite decimals, observing to increase the right-hand figure of the subtrahend by 1, if the lower repetend is greater than the upper. Make a repetend in the result similar and conterminous with those above.*

EXAMPLES FOR PRACTICE.

2. From $25.12\dot{7}$ take $14.\dot{6}\dot{5}$. Ans. $10.51\dot{1}\dot{2}$.
3. From $.\dot{1}0\dot{7}$ take $.0\dot{4}\dot{3}$. Ans. $.06\dot{3}67276\dot{3}$.

4. From $21.0\dot{4}$ take 15.6.25.
5. From $8.\dot{1}2\dot{3}$ take $5.\dot{2}\dot{9}\dot{2}$.
6. From $.\dot{5}7\dot{6}$ take .25. Ans. $.32\dot{6}5\dot{7}$.
7. From .576 take $.\dot{2}\dot{5}$. Ans. $.323\dot{4}\dot{7}$.
8. From $41.2\dot{3}2\dot{4}$ take $23.\dot{5}9\dot{4}$.

MULTIPLICATION OF REPEATING DECIMALS.

Art. 200. To multiply a repeating decimal.

Ex. 1. Multiply $9.\dot{2}5\dot{6}$ by 5.7.

FIRST METHOD.

$$\begin{array}{r} 9.\dot{2}56256256\dot{6} \\ 5.7 \\ \hline 64793793793 \\ 462812812813 \\ \hline \end{array}$$

$52.7606606606 = 52.7\dot{6}0\dot{6}$

SECOND METHOD.

$9.\dot{2}5\dot{6} = 9\frac{256}{999} : \times 5\frac{7}{10} = \frac{9247}{999} \times \frac{57}{10} : = \frac{527079}{9990} = 52.7\frac{606}{999} : = 57.7\dot{6}0\dot{6}$.

Rule.—*If the multiplier contains no repetend, first extend the repetend of the multiplicand enough to secure every figure of the repetend of the product, then multiply as if the multiplicand were a finite decimal.*

If the multiplier contains a repetend, reduce it to a common fraction, multiply, and reduce the result to the form of a repetend, as in addition of repetends; or reduce both factors to common fractions, and multiply, and reduce the result to a decimal.

EXAMPLES FOR PRACTICE.

2. Multiply $.\dot{6}\dot{3}$ by 4.3. Ans. $2.7\dot{3}\dot{6}$.
3. Multiply $.3\dot{7}$ by .421. Ans. $.1590\dot{4}$.
4. Multiply .534 by $.2\dot{5}\dot{7}$. Ans. $.137\dot{5}\dot{4}$.
5. Multiply $.4\dot{6}$ by $.\dot{7}\dot{2}$. Ans. $.3\dot{3}\dot{9}$.
6. Multiply $2.\dot{1}\dot{5}$ by $3.\dot{2}0\dot{4}$. Ans. $6.\dot{8}9\dot{3}$.

7. Multiply $.\dot{2}\dot{3}$ by 3.215: 5.128 by $.\dot{4}6$.
8. Multiply $.7\dot{5}$ by $.\dot{7}\dot{5}$: $.3\dot{4}$ by $.4\dot{3}$.
9. Multiply $.151\dot{2}$ by $.2\dot{3}\dot{5}$: $3.\dot{4}\dot{5}$ by $2.1\dot{6}$.

DIVISION OF REPEATING DECIMALS.

Art. 201. To divide a repeating decimal.

Ex. 1. Divide $7.2\dot{7}\dot{5}$ by 1.2.

FIRST METHOD.

$$1.2\,)\,\underline{7.275757575}$$
$$6.06313131 = 6.06\dot{3}\dot{1}$$

SECOND METHOD.

$7.2\dot{7}\dot{5} = 7.2\frac{75}{99}, = \frac{7203}{990}$: $\div\ 1\frac{2}{10} = \frac{7203}{990} \times \frac{10}{12} = \frac{7203}{1188} = 6.06\frac{31}{99} = 6.06\dot{3}\dot{1}$.

Rule.—*If the divisor contains no repetend, first extend the repetend of the dividend enough to secure every figure of the repetend of the quotient, and then divide as if the dividend were a finite decimal.*

If the divisor contains a repetend, reduce both repetends to common fractions, divide, and reduce the quotient to a decimal.

EXAMPLES FOR PRACTICE.

2. Divide $.3\dot{3}\dot{9}$ by $.\dot{7}\dot{2}$. Ans. $.4\dot{6}$.
3. Divide $.\dot{6}\dot{3}$ by $.\dot{1}$. Ans. $5.\dot{7}\dot{2}$.
4. Divide $2.7\dot{3}\dot{6}$ by 4.3. Ans. $.\dot{6}\dot{3}$.
5. Divide $.5\dot{1}\dot{2}$ by $.2\dot{6}$: 3.25 by $.1\dot{5}\dot{4}$.
6. Divide $.\dot{8}\dot{1}$ by $2.\dot{3}$: $3.\dot{7}\dot{8}$ by $1.\dot{1}\dot{8}$.
7. Divide $5.\dot{3}2\dot{4}$ by 2.5: $.\dot{4}\dot{5}$ by $.\dot{3}$.

MISCELLANEOUS EXAMPLES IN DECIMALS.

1. How much land in 4 fields containing 7.35 acres, 4.875 acres, 6.465 acres, and 10.7 acres, respectively?
2. A man owning a farm of 500 acres, sold to A, 80.5 acres,

to B, 100.45 acres, and to C, 90.75 acres. How many acres had he left?

3. There are 16.5 feet in a rod; how many feet are there in 60.48 rods?

4. At $\$.62\frac{1}{2}$ per bushel, how many barrels of apples, each containing $2\frac{3}{4}$ bushels, can be bought for $41.25? Ans. 24.

5. If 12 boxes of soap, each weighing 60 pounds, can be bought for $46.80, what is the cost per pound? Ans. $.065.

6. Divide 21.6 by $2\frac{1}{4}$ tenths. Ans. 96.

7. If .75 tons of hay cost $20.25, what is a ton worth?

8. If 1 ton of hay is worth $20.25, what is .6 ton worth?

9. A stock-dealer bought 80 head of sheep at $4.75, and 64 head at $5.5 a head; at what price per head must he sell them to gain $116 on the whole? Ans. $5.75.

10. Reduce $\dfrac{1\frac{7}{8}}{3\frac{1}{5}}$ to a decimal.

11. Reduce .0875 to a common fraction.

12. A miller wished to buy an equal number of bushels of corn, rye, and wheat; the corn cost $\$.68\frac{1}{4}$, the rye $\$.93\frac{3}{4}$, and the wheat $\$1.87\frac{1}{2}$ per bushel. How many bushels of each could he buy for $2264.76? Ans. 648.

13. If 25 barrels of flour cost $256.25, what will 33 barrels cost?

14. A grocer bought 124.5 pounds of butter at $\$.15\frac{3}{4}$ per pound, $\frac{4}{5}$ of which he sold at $.1875 a pound, and the rest at cost; how much did he gain?

15. From an oil tank containing 4325 gallons, 81.5 barrels, of 42.25 gallons each, were drawn off. How many gallons remained?

16. There are 5.5 yards in one rod, and 320 rods in a mile; how many yards in a mile?

17. A man sold 10 bushels of onions at $2.75 a bushel, and received in payment 30 pounds of coffee, at $.28 a pound, 50 pounds of sugar, at $.14 a pound, 20 pounds of cheese, at $.165 a pound, and the balance in molasses, at $.80 a gallon; *how many* gallons did he receive?

18. Bought 12.5 barrels of apples for \$34.37½, and sold them at a profit of \$$.87\dot{5}$ per barrel; at what price per barrel were they sold? Ans. \$3.625.

19. A drover sold 80 head of sheep for \$518.25, thereby gaining \$88.25; what was the cost per head? Ans. \$5.375.

20. Divide 2¾ by .64.

21. A and B start from the same place at the same time, and travel in opposite directions, A traveling at the rate of 22⅘ miles per day, and B at the rate of 24.64 miles per day. How far apart will they be at the end of 12.45 days?

Ans. 581.14.

22. A man expended \$181.35 in the purchase of wheat at \$1.65 a bushel, rye at \$1.05 a bushel, corn at \$.70 a bushel, and oats at \$.63 a bushel, buying the same quantity of each kind; how many bushels did he buy in all? Ans. 180.

23. There are 16.5 feet in a rod, and 320 rods in a mile. How many feet in a mile? Ans. 5280.

24. If the circumference of the forward wheel of a carriage is 13.75 feet, and the circumference of the hind wheel is 15.25 feet. How many times will each revolve, and how many more times will one revolve than the other, in moving 7⅝ miles? Ans. 2928, 2640, and 288.

25. If each stroke of the piston-rod of a locomotive will move a train of cars 13.75 feet, how many strokes will be required to move the train a distance of 81.5 miles?

26. If 75 bushels of apples cost \$41.25, how many bushels can be bought for \$110? Ans. 200.

SYNOPSIS OF DECIMAL FRACTIONS.

KINDS.	KINDS.	KINDS.	KINDS.	OPERATIONS.
Finite.				Notation.
Infinite.	Repeating.	Pure.	Similar.	Numeration.
		Mixed.	Dissimilar.	Reduction.
			Conterminous.	Addition.
	Non-repeating.			Subtraction.
				Multiplication.
				Division.

CHAPTER XI.

DENOMINATE NUMBERS.

Art. 202. A **denominate number** is a concrete number expressed in units which measure magnitude. (See Arts. 4–8.)

A **denomination** is a measuring unit of a certain kind or value. Thus, in measuring time, we may use measuring units of various values, such as *centuries, years, months, weeks, days, hours, minutes,* and *seconds,* and these units would be called *denominations of time.*

All true measuring units of the same name, or denomination, must have the same value, and it must not change. Hence such units must be *arbitrary,* (See Art. 5,) since arbitrary units can be clearly and accurately defined, and made permanent. Most natural units, because they are variable in quantity, even when of the same name, and changeable from time to time, cannot be taken as denominations of a denominate number. (See Art. 8.)

A **standard** of measure is a measuring unit whose multiples or parts form other denominations in that kind of measure.

Art. 203. Denominate numbers express quantity of seven different kinds; namely, *length, surface, volume, weight, divergence* or *angle, time,* and *money* or *value in trade.*

The **measures of extension** are those of length, surface, and volume. (See Art. 4.) They are also called *measures of space.*

Measures of weight may more properly be called *measures of force,* because weight is only the measured force of gravitation, and because units of weight are used to measure other forces.

Measures of time may more properly be called *measures of duration,* because time is more properly measured duration.

Measures of multitude only are either abstract or concrete units, which do not always express the same value, or the same quantity, of that which makes the unit. Thus, the abstract unit *1* merely signifies *unity of number,* without implying quantity of any constituents, and the concrete unit *1 tree* merely signifies a *single tree,* without implying its size or quantity of constituents. Measures of multitude only are not denominate. (See Art. 8.)

MEASURES OF EXTENSION, OR OF SPACE.

Art. 204. The measures of extension are of three kinds, namely, those of *distance,* or *length,* those of *surface,* or *area,* and those of *volume,* or *bulk,* or *capacity.*

I.

LINEAR MEASURE.

Art. 205. Linear Measure is measure of distance, or length. It is sometimes called *Long Measure.*

Different nations use different units of linear measure, and the different occupations in a nation often use units of measure peculiar to themselves. Some of these systems of units have a regular *scale of relative value,* (See Art. 26,) and some are irregular.

ENGLISH AND UNITED STATES LINEAR MEASURE.

1. COMMON TABLE.

Irregular scale of values.

12	inches, (marked in.,)	make	1 foot,	ft.
3	feet,	"	1 yard,	yd.
5½	yards, or 16½ feet,	"	1 rod,	rd.
40	rods,	"	1 furlong,	fur.
8	furlongs, or 320 rods,	"	1 statute mile,	mi.
3	miles,	"	1 league,	lea.

UNIT EQUIVALENTS.

									ft.		in.
							yd.		1	=	12
					rd.		1	=	3	=	36
			fur.		1	=	$5\frac{1}{2}$	=	$16\frac{1}{2}$	=	198
mi.			1	=	40	=	220	=	660	=	7920
1	=		8	=	320	=	1760	=	5280	=	63360

NOTE. 1.—The standard of linear measure is the *yard*. Its length is determined as follows:—Divide into 391393 equal parts the length of a pendulum which beats seconds of mean time in London, on a level with the sea, in a vacuum; the yard is 360000 of these parts, and the inch is 10000 of such parts. Hence such a pendulum is 39.1393 inches long.

NOTE 2.—The inch is subdivided to suit the convenience of different occupations. The more common subdivisions are halves, fourths, eighths, sixteenths, thirty-seconds, &c. Twelfths of an inch are often used, and are called *lines*.

Decimal divisions of an inch, and of parts of an inch, are now very common in instruments for drawing, surveying, &c.

NOTE 3.—Dealers in cloth once subdivided a yard into quarters, and quarters into quarters, called *nails;* also used a denomination called an *ell*, forming a peculiar table of cloth measure, thus:—

TABLE OF CLOTH MEASURE.

$2\frac{1}{4}$ inches, (in.,)	make	1 nail,	na.
4 nails, or 9 inches,	"	1 quarter,	qr.
4 quarters,	"	1 yard,	yd.
3 quarters,	"	1 Flemish ell.	
5 quarters,	"	1 English ell.	
6 quarters,	"	1 French ell.	

The peculiar denominations of this table are seldom used, the more common divisions of the yard being halves, quarters, eighths, and sixteenths.

NOTE 4.—The statute mile is 5280 feet, or 1760 yards. A *nautical mile*, or *geographic mile*, used by mariners and in geographical and astronomical calculations, is 1.15 times a statute mile.

NOTE 5.—Carpenters, plasterers, and other artisans, sometimes subdivide the foot *duodecimally*, thus:—

DUODECIMAL TABLE.

12 inches, or primes, marked	′	make	1 foot,	ft.
12 seconds,	" ″	"	1 inch,	
12 thirds,	" ‴	"	1 second,	
12 fourths,	" ⁗	"	1 third, &c.	

Miscellaneous Linear Units.

Art. 206. Persons in different occupations naturally assume units of measure convenient to them. These are frequently derived from the dimensions of the body, because grasping, applying the palm of the hand, stretching the fingers over objects, stretching the arms over longer objects, striding over distances, &c., are convenient approximate measurements, in the absence of accurate ones. Though the sizes of bodies differ, yet averages for rough estimates are somewhat as follows:—

Miscellaneous Table.

3 inches are 1 palm, (breadth of the four fingers.)
4 inches are 1 hand, (breadth of palm and thumb, used in measuring the height of horses over the fore feet.)
9 inches are 1 span, (distance from end of thumb to end of little finger in the extended hand.)
18 inches are 1 English cubit, (distance from elbow to end of middle finger.)
21.888 inches are 1 Hebrew cubit, or cubit of the Bible.
5 feet are 1 pace of two steps, that is, the distance between two prints of the same foot.
3 feet are 1 pace, that is, a long stride.
6 feet, or 8 spans, are 1 fathom, (the outstretch of both arms, used by mariners.)
120 fathoms are 1 cable's length, (used by mariners.)

Note 1.—The term *cable's length* is derived from the fact that mariners used the ordinary lengths of ship's cables as a measure for hasty estimates.

Note 2.—The *knot*, a denomination in mariners' language, is a measure of the velocity of vessels. A line, called a *log-line*, to which a float, called a *log*, is attached, runs freely over the stern as the vessel sails. Knots of colored cloth are attached at intervals, and the number of these knots passing in a half-minute is the number of nautical miles the vessel sails per hour. The space of a knot is about $50\frac{2}{3}$ feet.

2. Engineers' and Surveyors' Table.

Art. 207. Engineers and surveyors use a kind of linear measure in surveying roads, canals, boundaries, &c., which is called *chain measure*, from the fact that a chain of peculiar construction is frequently used by them in measuring.

Table.

Irregular scale of values.

$7\frac{92}{100}$ inches, (in.,)	make	1 link,	l.
25 links,	"	1 pole, or rod,	p.
100 links, or 4 poles, or 66 feet,	"	1 chain,	ch.
10 chains,	"	1 furlong,	fur.
8 furlongs, or 80 chains,	"	1 mile,	mi.

Unit Equivalents.

						l.		in.
				p.		1	=	7.92
		ch.		1	=	25	=	198
mi.		1	=	4	=	100	=	792
1	=	80	=	320	=	8000	=	63360

Note 1.—A surveyor's chain is called *Gunter's chain*, from the name of its inventor. It is usually composed of stout wire, in 100 equal links, with marks to indicate rods, and has stout handles at the ends for straightening, carrying, and applying the chain as a measure.

Note 2.—Civil engineers sometimes use a chain of 100 links, each of which is 1 foot long.

Note 3.—Distances are usually expressed by surveyors in chains and links merely, and, because links are hundredths of a chain, they are often written as decimal fractions. Thus 579 chains 83 links may be written 579.83 ch.

3. Metric Linear Measure.

Art. 208. The irregular and troublesome measures in use were attended with so much inconvenience that the *French* Government, in 1795, adopted a system with a per-

fectly regular decimal scale, and with a regular system of names for the different denominations. The standard of this system is the *metre*. This name is pronounced, in English, *meet'-er;* in French, *mā'-tr*.

A **metre** is a ten-millionth part of the distance from the Equator to the North Pole, on the meridian of Paris. Its length is 39.37 inches.

NOTE.—A survey was made by Delambre and Mechain, in 1792–1794, of the distance from Dunkirk, France, to Barcelona, Spain, and the latitudes of these two places were determined. From these elements the distance from the Equator to the North Pole was computed. In 1795 the French Government ordained that a ten-millionth part of this computed distance should be called a *metre*, and should be the unit of length. The length of a metre was marked on a bar of platinum so that the marked distance would be the standard at the temperature of melting ice, and this bar was deposited in the archives of France for reference. From the metre all units of measure and weight were established. This Metric System has since been steadily gaining in the world. Many of the governments of Europe have adopted it. In 1864 the English Government permitted its use, and the Government of the United States, by an Act approved July 28, 1866, authorized contracting parties to use its denominations, and provided a schedule of equivalent values, which may be lawfully used for reducing metric to customary weights and measures. These values are given in all the metric tables of this chapter.

Art. 209. Most of the names of the denominations of the Metric System are formed on a regular plan, so as to be significant of the value of the denominations.

In this plan, the name of that denomination which is

$\frac{1}{10}$ of the unit, has the Latin prefix *deci, (dess'-e)*, meaning *tenth.*

$\frac{1}{100}$ of the unit, has the Latin prefix *centi, (sent'-e)*, meaning *hundredth.*

$\frac{1}{1000}$ of the unit, has the Latin prefix *milli, (mill'-e)*, meaning *thousandth.*

10 times the unit, has the Greek prefix *deca, (dek'-a)*, meaning *ten.*

100 times the unit, has the Greek prefix *hecto*, (*hek'-to*), meaning *hundred*.

1000 times the unit, has the Greek prefix *kilo*, (*kil'-o*), meaning *thousand*.

10,000 times the unit, has the Greek prefix *myria*, (*mir'-e-a*), meaning *ten thousand*.

TABLE.

Regular scale of values, 10.

10 **metres**, (marked M.,)	make	1 decametre,	marked	D. M.
10 decametres,	"	1 hectometre,	"	H. M.
10 hectometres,	"	1 kilometre,	"	K. M.
10 kilometres,	"	1 myriametre,	"	M. M.
$\frac{1}{10}$ of a **metre**,	is	1 decimetre,	"	d. m.
$\frac{1}{10}$ of a decimetre,	"	1 centimetre,	"	c. m.
$\frac{1}{10}$ of a centimetre,	"	1 millimetre,	"	m. m.

NOTE 1.—The names of this table are pronounced, in English, as follows:—*dek'-a-meet'-er, hek'-to-meet'-er, kil'-o-meet'-er, mir'-e-a-meet'-er, dess'-e-meet'-er, cent'-e-meet'-er, mil'-le-meet'-er.*

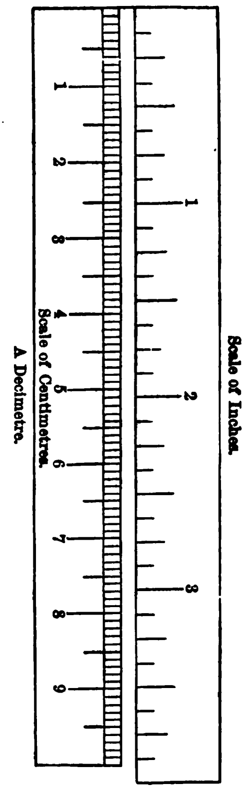
NOTE 2.—The schedule of equivalents for measures of length, adopted by Congress, is as follows:—

1 myriametre = 6.2137 miles.	1 metre = 39.37 inches.
1 kilometre = 0.62137 " or 3280 ft. 10 in.	1 decimetre = 3.937 "
1 hectometre = 328 ft. 1 in.	1 centimetre = 0.3937 "
1 decametre = 393.7 inches.	1 millimetre = 0.0394 "

NOTE 3.—The learner can assist his appreciation of the length of a metre by marking off 39⅜ inches. This differs but slightly from a metre. A decimetre is shown on the next page.

NOTE 4.—Some of the denominations of this table are little used. Considerable distances are generally expressed in *metres* or *kilometres;* sometimes very great distances in *myriametres.* It is customary to express hundreds of metres as such rather than as *hectometres.* For example, 5 *hectometres* would generally be expressed as *500 metres.* Again, 5 *decametres* would generally be expressed as *50 metres;* and 4 *myriametres* would be expressed as *40 kilometres.* It is plain that the values are the same in these preferred expressions, and convenience in *estimating* determines the selections.

NOTE 5.—The figure in the margin represents a decimetre, with its subdivision into 10 centimetres and each centimetre into 10 millimetres, placed by the side of a scale of 4 inches, each inch being divided into 4 quarters, and each quarter into two-eighths of an inch.



NOTATION.

Art. 210. The French indicate a denomination by placing its initial letter or letters as an index to its units. For example, they write *9 metres 257 millimetres* thus: $9^{m},257$. In this country other modes have also been practiced, such as placing the initial letter of the denomination before the number, or after it. Thus, the above number may be written M. 9.257, or 9.257 M.

NUMERATION.

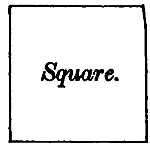
Art. 211. To read a mixed number in metric linear measure, either *read it as a mixed decimal of its denomination*, or *read the integral part as a whole number of its denomination, and the fractional part either as a decimal of that denomination, or as a whole number of its lowest denomination.*

ILLUSTRATION.—The number given in Art. 210 may be read *9 metres 257 millimetres*, or *9 metres and 257 thousandths.*

II.

MEASURE OF SURFACE.

Art. 212. **Surface Measure** is measure of quantity of surface. It is measured by the product of two dimensions at right angles to each other. It is often called *square measure*, because the units used in expressions of surface are *squares.*



A **square** is a plane surface having four equal sides and four equal angles, called *right angles.*

A square takes its name from the length of its side. Thus, a *square inch* is a square, each of whose sides is an inch long; a *square foot* is a square, each of whose sides is a foot long, &c.

A **rectangle** is *any* four-sided surface whose angles are equal. A **square** is one kind.

Area, or **superficies,** is quantity of surface; also, a surface inclosed by lines is called *an area.*

Art. 213. Any rectangle contains as many square units of a certain name as the product of the linear units of the same name in its length multiplied by those in its breadth.

ILLUSTRATIONS.

Figure 1.

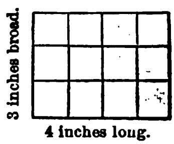
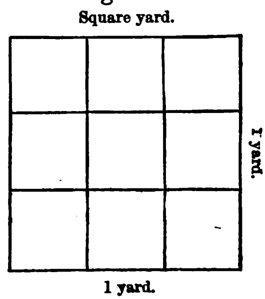


Figure 2.



4 in. × 3 in. = 12 square inches.

3 ft. × 3 ft. = 9 square feet.
1 yd. × 1 yd. = 1 square yard.

Figure 1 represents a surface 4 inches long and 3 inches broad, and contains three rows of four squares each, or four rows of three squares each, in all 12 squares, representing 12 square inches.

Figure 2 represents a surface 1 yard, or 3 feet, long, and of the same breadth, and contains 3 rows of 3 squares each, in all 9 squares, representing 9 square feet. It also represents a surface 1 yard long and 1 yard broad, containing 1 square yard.

Hence, a surface *1 unit square* contains *1 square unit of the same name;* a surface *2 units square* contains *4 square units of the same name;* a surface *3 units square* contains *9 square units of the same name;* &c.

NOTE.—Every multiplier is an abstract number, because it merely indicates how many expressions of the multiplicand are required. Literally it is absurd to say "*3 feet times 3 feet*," and that is not our mental process in obtaining area. Neither do we mean "*3 times 3 linear feet are 9 square feet*," for that is untrue, because 3 times 3 linear feet are 9 linear feet. We mean merely that *area* contains as many *units of area* as there are abstract units in the product of the *number* of units of length by the *number* of units of breadth.

ENGLISH AND UNITED STATES SURFACE MEASURE.

1. COMMON TABLE.

Irregular scale of values.

144	square inches, (sq. in.,)	make	1 square foot,	sq. ft.
9	square feet,	"	1 square yard,	sq. yd.
30¼ / 272¼	square yards, or square feet,	"	1 sq. rd., pole, or perch,	P.
40	square rods, or perches,	"	1 rood,	R.
4	roods, or 160 sq. rds.,	"	1 acre,	A.
640	acres,	"	1 square mile,	sq. mi.

UNIT EQUIVALENTS.

sq. mi.	A.	R.	sq. rd.	sq. yd.	sq. ft.	sq. in.
					1 =	144
				1 =	9 =	1296
			1 =	30¼ =	272¼ =	39204
		1 =	40 =	1210 =	10890 =	1568160
	1 =	4 =	160 =	4840 =	43560 =	6272640
1 =	640 =	2560 =	102400 =	3097600 =	27878400 =	4014489600

NOTE 1.—Let the pupil illustrate upon the slate or blackboard the statements of the table.

NOTE 2.—Although the parties to any contract of work can estimate that work in any units that suit their convenience, yet artisans are accustomed to estimate their work mostly as follows:—

1. Glazing and stone-cutting, by the — square foot.
2. Painting, paving, plastering, ceiling, and paper-hanging, by the — square yard.
3. Flooring, partitioning, roofing, by the *square*, of 10 ft. square, or — 100 square feet.
4. Bricklaying is generally estimated by the thousand bricks, but sometimes by the *square yard*, or the *square of 100 square feet*, provided that the work is 1 foot, or 1½ bricks thick.

5. In measuring irregular work, such as mouldings, cornices, and curved surfaces in general, artisans seek to estimate the whole surface. This implies that the quantity of that surface is obtained in the units of square measure, which are *flat*. This is found in mouldings which are straight by making the measuring line follow all the irregularities *crosswise*, for one dimension, and the *length* is the other. In *spherical* and other curves, the surface is found by the rules of mensuration.

2. Surveyors' Surface Measure.

Art. 214. Surveyors' Square Measure is used in computing the area of land, or other surveyed surfaces.

Table.

Irregular scale of values.

62.7264	square inches,	make	1 square link.	sq. l.
625	square links,	"	1 pole,	P.
16	poles,	"	1 square chain,	sq. ch.
10	square chains,	"	1 acre,	A.
640	acres,	"	1 square mile,	sq. mi.

Unit Equivalents.

sq. mi.	A.	sq. ch.	P.	sq. l.	sq. in.
				1 =	62.7264
			1 =	625 =	39204
		1 =	16 =	10000 =	627264
	1 =	10 =	160 =	100000 =	6272640
1 =	640 =	6400 =	102400 =	64000000 =	4014489600

Note.—This table is based on the supposition that the chain used is Gunter's chain. If another chain is used, a table adapted to it may be easily constructed, by Art. 213.

3. Metric Surface Measure.

Art. 215. The standard of surface measure, for large surfaces, in the Metric System, is the *are*, (pronounced *air.*) It equals a *square decametre*, that is, a square surface, each of whose sides is 10 metres long, and is, therefore, equal to 100 square metres. It is mostly used in expressing the surface of land and water.

TABLE.

Regular scale of values, 100.

100 centares, (marked c.a.,)	make	1 **are,**	marked	A.
100 ares,	"	1 hectare,	"	H.A.

UNIT EQUIVALENTS.

						c.a. or sq. M.		sq. yd.		sq. in.
			A.			1	=	1.196	=	1550
H.A.			1	=		100	=	119.6	=	155000
1	=		100	=		10000	=	2.471 acres.		

NOTE 1.—*Centare* (pronounced cent′air) is sometimes written *centiare*, (pronounced cent′e air.)

NOTE 2.—The denominations *milliare*, *deciare*, *decare*, *kilare*, and *myriare* are not used. A *centare* being a square metre, an *are* can be represented by a square whose side is 10 metres long, and a *hectare* by a square whose side is 10 times 10 metres, that is, 100 metres long. But a *deciare*, which would be 10 square metres of surface, *cannot be represented by a square whose side is a whole number of metres.* The same is true of a *decare*, a surface of 1000 square metres, and of a *kilare*, a surface of 100,000 square metres.

NOTE 3.—For general purposes it is customary to express surface by the square of some denomination of linear measure.

TABLE.

Regular scale of values, 100.

100 **sq. metres,** (M^2)	equal	1 sq. decametre,	marked	$D.M.^2$
100 sq. decametres,	"	1 sq. hectometre,	"	$H.M.^2$
100 sq. hectometres,	"	1 sq. kilometre,	"	$K.M.^2$
100 sq. kilometres,	"	1 sq. myriametre,	"	$M.M.^2$
$\frac{1}{100}$ **sq. metre,**	equals	1 sq. decimetre,	"	$d.m.^2$
$\frac{1}{100}$ sq. decimetre,	"	1 sq. centimetre,	"	$c.m.^2$
$\frac{1}{100}$ sq. centimetre,	"	1 sq. millimetre,	"	$m.m.^2$

EQUIVALENTS.

1 $M.M.^2$ =	24711.43 acres.		1 $M.^2$ =	1550	sq. in.
1 $K.M.^2$ =	247.1 "		1 $d.m.^2$ =	15.5	"
1 $H.M.^2$ =	2.471 "		1 $c.m.^2$ =	.155	"
1 $D.M.^2$ =	119.6 sq. yds.		1 $m.m.^2$ =	.00155	"

NUMERATION.

Art. 216. To read a mixed number in metric square measure, *either read it as a mixed decimal of its denomination, or read the integral part as a whole number of its denomination, and the fractional part either as a decimal of that denomination, or as a whole number of its lowest denomination.*

NOTE.—Remember that each denomination occupies *two* decimal places.

ILLUSTRATION.—The number 5.604237K.M.[2] may be read as *5 square kilometres and 604237 millionths*, or *5 square kilometres and 604237 square metres.*

III.

MEASURE OF VOLUME.

Art. 217. The **volume** of any thing is the quantity of space which it occupies. It is measured by the product of three dimensions at right angles to each other. It is often called **solidity**, or **content**, or **bulk.**

The measure of volume is often called **solid measure,** in reference to *solidity*, the thing measured. It is also called **cubic measure,** because the units used in expressions of volume are *cubes.*

A **cube** is a body bounded by six squares. Therefore its length, breadth, and thickness are equal.

A cube takes its name from the length of one of its edges. Thus, a *cubic inch* is a cube each of whose edges is an inch long, or whose sides are square inches; a *cubic foot* is a cube each of whose edges is a foot long, or whose sides are each a square foot, &c.

Art. 218. Any six-sided body, whose sides are rectangles,* contains as many cubic units of a certain name as the product of the linear units of the same name in its length, by those in its breadth, by those in its thickness.

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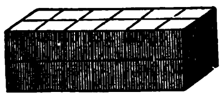
* Such a body is called a *rectangular parallelopipedon.*

ILLUSTRATIONS.

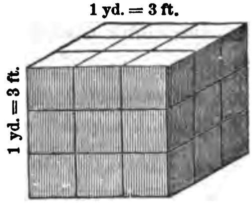
Figure 1.



6in. × 2in. × 2in. = 24 cu. in.

Figure 2.

1 yd. = 3 ft.



1 yd. × 1 yd. × 1 yd. = 1 cu. yd., and 3 ft. × 3 ft. × 3 ft. = 27 cu. ft.

Figure 1 represents a rectangular six-sided body, 6 inches long, 2 inches broad, and 2 inches high. Therefore one row of cubes represents 6 cubic inches, the bottom layer of 2 rows 2 × 6 = 12 cubic inches, and 2 layers 2 × 12 = 24 cubic inches.

Figure 2 represents a cubic yard, 3 feet long, 3 feet broad, and 3 feet high. Therefore one row of cubes represents 3 cubic feet, the bottom layer of 3 rows 3 × 3 = 9 cubic feet, and 3 layers 3 × 9 = 27 cubic feet.

ENGLISH AND UNITED STATES CUBIC MEASURE.

1. COMMON TABLE.

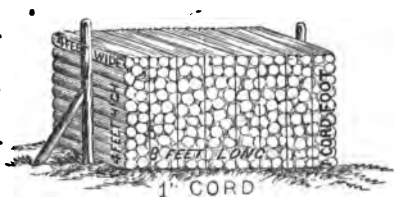
Irregular scale of values.

1728 cubic inches, (cu. in.,)	make	1 cubic foot,	cu. ft.
27 cubic feet,	"	1 cubic yard,	cu. yd.

UNIT EQUIVALENTS.

cu. yd.		cu. ft.		cu. in.
		1	=	1728
1	=	27	=	46656

NOTE 1.—A *cord of wood* is 128 cubic feet. A pile of wood 8 ft. long, 4 ft. wide, and 4 ft. high, contains a cord, because 8 ft. × 4 ft. × 4 feet = 128 cu. ft. A *cord-foot* is one foot in length of such a pile, being 1 ft. long, 4 ft. wide, and 4 ft. high, equal to 16 cu. ft. Hence 8 cord feet make a cord.



NOTE 2.—A *foot* in *board measure* is 1 ft. long, 1 ft. broad, and 1 inch thick; that is, it is *a foot square of an inch board.*

Board stuff less than an inch thick is commonly reckoned as 1 inch thick, in finding its number of feet.

Lumber more than an inch thick is reckoned as if made up of inch boards. Thus, a piece of timber 6 inches thick is reckoned as 6 boards, each 1 inch thick, and of the same length and breadth as the piece.

Fifty cubic feet were formerly reckoned a *load*, or *ton*, of timber, and it was reckoned that the *round* timber which, on squaring, would yield 40 cu. ft. was as much of a *load*, or *ton of carriage*, as 50 cu. ft.

NOTE 3.—A *perch of masonry* is $24\frac{3}{4}$ cu. ft. It is conveniently represented by a mass 1 rod ($16\frac{1}{2}$ ft.) long, 1 ft. high, and $1\frac{1}{2}$ ft. thick. In estimating the masonry in the walls of buildings, the *length outside*, or *girth*, is taken as the length, no allowance being made for corners.

A brick which is 8 inches long, 4 broad, and 2 thick, contains 64 cubic inches. A cubic foot of such bricks, piled *solid*, without cement, contains 27 bricks; with cement, usually about $22\frac{1}{2}$ bricks. Five courses of such bricks usually make 1 ft. of height of wall.

NOTE 4.—Excavations are usually made by the *cubic yard*, and that amount of earth is called a *load*.

NOTE 5.—Transportation of light and bulky articles is usually charged by the *cubic foot*, and that of heavy articles by actual weight.

2. METRIC SOLID, OR CUBIC MEASURE.

Art. 219. The standard of solid or cubic measure in the Metric System is the **cubic metre.** Its multiples and divisions have not received the regular names of the Metric System. (See Art. 209.) It equals 1.308 cu. yd., or 35.316 cu. ft.

When the cubic metre is applied to measure the volume of firewood and building timber, it is called the **stere,** (pronounced *stair.*) In this case the denominations *decastere* and *decistere* are sometimes used.

TABLE.

Regular scale of values, 1000.

$\frac{1}{1000}$ **cubic M.** (M^3)	equals	1 cubic d.m.,	marked	$d.m.^3$
$\frac{1}{1000}$ cubic d.m.,	"	1 cubic c.m.,	"	$c.m.^3$
$\frac{1}{1000}$ cubic c.m.,	"	1 cubic m.m.,	"	$m.m.^3$

Also, for timber and firewood—

10 steres, (S.)	equals	1 decastere,	marked	D.S.
$\frac{1}{10}$ **stere,**	"	1 decistere,	"	d.s.

NOTE 1.—It is plain that great volumes can be expressed in cubic *decametres*, hectometres, kilometres, or myriametres.

NUMERATION.

Art. 220. In a mixed decimal whose integral part is cubic metres, or *steres*, the *third* decimal place must be *cubic decimetres*, the *sixth* must be *cubic centimetres*, and the *ninth* must be *cubic millimetres*. It would generally be preferable to read the decimal part as a decimal.

3. MEASURE OF CAPACITY.

Art. 221. **Capacity** is power of containing, or holding. It is volume, considered as space to contain whatever can occupy it. Its measure is, therefore, an application of the measure of volume.

ENGLISH AND U. S. MEASURES OF CAPACITY.

Art. 222. In reference to their application, measures of capacity are either *liquid*, or *dry*.

Liquid Measure is measure of the quantity of liquids, or of the vessels which contain them.

Dry Measure is measure of the quantity of solid matter in a somewhat divided state; such as fruit, grain, salt, ashes, coal, lime, &c.

1. COMMON LIQUID MEASURE.

TABLE.

Irregular scale of values.

4 gills, (gi.)	make	1 pint,	pt.
2 pints,	"	1 quart,	qt.
4 quarts,	"	1 gallon,	gal.

UNIT EQUIVALENTS.

				pt.		gi.
		qt.		1	=	4
gal.		1	=	2	=	8
1	=	4	=	8	=	32

NOTE 1.—The standard of liquid measure in the United States is the *gallon*, whose volume is 231 cubic inches. It is the old English Winchester Wine Gallon. It holds 8.3388 pounds avoirdupois of distilled water at 39$\frac{83}{100}$ degrees of Fahrenheit's thermometer, when the barometer is at 30 inches. It is called the Winchester Wine Gallon from the fact that the standard measure was kept at Winchester, England. That standard is now abolished in England. Since 1836 the standard of liquid measure in Great Britain has been a gallon, whose volume is 277.274 cubic inches. It holds 10 pounds avoirdupois of distilled water at 62 degrees, Fahrenheit, when the barometer is at 30 inches. It is called the *Imperial Gallon*.

NOTE 2.—Formerly, milk and malt liquors were measured by a gallon containing 282 cubic inches, but this has gone out of use.

NOTE 3.—Casks for containing liquids are made of any convenient size, and their contents are *gauged*, or measured, before sale. The cask now popularly called a *barrel* has a capacity of about forty gallons, though commonly varying more or less above or below that capacity. So, also, larger casks, which are called *hogsheads*, *pipes*, &c., are of almost any desired capacity. The laws of some States define what shall be considered a barrel and hogshead, if contracting parties do not otherwise fix the meaning of those denominations. Such laws generally state the following values:—31½ gallons are 1 barrel; 2 barrels, or 63 gallons, are 1 hogshead; 2 hogsheads are 1 pipe; 2 pipes are 1 tun. Practically, these definitions are little regarded.

2. APOTHECARIES' FLUID MEASURE.

Art. 223. In compounding medical prescriptions, apothecaries use a subdivision of the Winchester Wine Gallon of 231 cubic inches.

TABLE.

Irregular scale of values.

60 minims, (♏)	make	1 fluidrachm,	fʒ.
8 fluidrachms,	"	1 fluidounce,	f℥.
16 fluidounces,	"	1 pint,	O.
8 pints,	"	1 gallon,	cong.

UNIT EQUIVALENTS.

						fʒ.		♏
				f℥.		1	=	60
		O.		1	=	8	=	480
Cong.		1	=	16	=	128	=	7680
1	=	8	=	128	=	784	=	47040

NOTE 1.—O. signifies *octarius*, or one-eighth of a gallon, and *cong.* is for *congius*, the Latin for a gallon.

NOTE 2.—A number of any denomination in United States liquid measure can be reduced to a like denomination of British liquid measure, (Imperial,) by dividing it by the decimal .83311.

Art. 224. 3. COMMON DRY MEASURE.

TABLE.

Irregular scale of values.

2 pints, (pt.)	make	1 quart,	qt.
8 quarts,	"	1 peck,	pk.
4 pecks,	"	1 bushel,	bu.

UNIT EQUIVALENTS.

				qt.		pt.
		pk.		1	=	2
bu.		1	=	8	=	16
1	=	4	=	32	=	64

NOTE 1.—The standard of dry measure in the United States is the *bushel*, which contains 2150.42 cubic inches. It is the old English, or Winchester bushel. It is conveniently represented by a cylindrical vessel 18½ inches in diameter, and 8 inches deep. It contains 77.627413 pounds avoirdupois of distilled water, at 39.83 degrees of Fahrenheit's thermometer, when the barometer is at 30 inches.

NOTE 2.—The present standard of dry measure in Great Britain is the bushel, which contains 2218.192 cubic inches. It holds 8 Imperial gallons, or 80 pounds avoirdupois of distilled water at 62 degrees, Fahrenheit, when the barometer is at 30 inches. It is called the Imperial Bushel. The English gallon is the same in liquid and dry measure. In England, 4 bushels are 1 *coomb*, and 2 coombs, or 8 bushels, are 1 *quarter*, so called because, at the English standard weight of a bushel of wheat, which is 70 pounds avoirdupois, 8 bushels are 560 pounds, which is one-quarter of the *long ton*, 2240 pounds. (See Art. 232, Note 2.)

NOTE 3.—The laws of most States define the number of pounds avoirdupois which shall be considered a bushel of various grains, fruits, &c. (See table under Avoirdupois Weight, Art. 232.)

Comparison of Common Dry and Liquid Measures.

Cubic inches in a	Gallon.	Quart.	Pint.	Gill.	Bushel.	Peck.
U. S. Liquid.......	231	$57\frac{3}{4}$	$28\frac{7}{8}$	$7\frac{7}{32}$		
U. S. Dry..........	$268\frac{4}{5}$	$67\frac{1}{5}$	$33\frac{3}{5}$	$8\frac{2}{5}$		
Eng. Dry & Liquid	$277\frac{274}{1000}$	$69\frac{3185}{10000}$	$34\frac{659}{1000}$	$8\frac{665}{1000}$		
U. S. Bushel.......					$2150\frac{42}{100}$	$537\frac{6}{10}$
English Bushel...					$2218\frac{192}{1000}$	$554\frac{548}{1000}$

4. Metric Measure of Capacity.

Art. 225. The standard of the measure of capacity in the Metric System is the *litre,* (pronounced *leet'-er.*) It is equal to a cubic decimetre.

Table.

Regular scale of values, 10.

10 **litres,** (L.)	equal	1 decalitre,	marked	D. L.
10 decalitres,	"	1 hectolitre,	"	H. L.
10 hectolitres,	"	1 kilolitre,	"	K. L.
$\frac{1}{10}$ **litre,**	equals	1 decilitre,	"	d. l.
$\frac{1}{10}$ decilitre,	"	1 centilitre,	"	c. l.
$\frac{1}{10}$ centilitre,	"	1 millilitre,	"	m. l.

NOTE 1.—The *litre* equals 61.02 + cubic inches, or 1.0567 quarts U. S. liquid measure, or 0.908 quarts U. S. dry measure.

NOTE 2.—The *kilolitre* is equal to the *stere,* or cubic metre. It contains 264.17 U. S. liquid gallons; the *hectolitre* 26.417 gallons; the *decalitre* 2.6417 gallons. The hectolitre contains 2 bushels 3.35 pecks, U. S. dry measure.

NOTE 3.—In this measure every decimal figure represents a denomination.

IV.

Measures of Weight, or of Force.

Art. 226. The **weight** of any terrestrial quantity of matter is the measured amount of its gravitating force toward *the centre of* gravity of the Earth. Weight, in its ultimate

sense, is the amount of the gravitating force of one quantity of matter toward another. In this sense it is applied to the heavenly bodies, and to any portions of matter gravitating toward any of them.

Art. 227. Under similar circumstances weight is in direct proportion to, or varies directly as, the quantity of matter. Hence, weight is a more perfect test of quantity of matter than are the measures of extension.

Weighing a portion of matter is comparing its gravitating force with that of another portion assumed as a standard.

Art. 228. The units of weight are, also, generally used to express the quantity of other forces, such as *pressure*, or driving force, and *traction*, or drawing force.

ENGLISH AND UNITED STATES WEIGHT.

Art. 229. The common measures of weight in England and the United States are of three scales, each irregular, namely, Troy Weight, Apothecaries' Weight, and Avoirdupois Weight.

1. Troy Weight.

Art. 230. **Troy Weight** is used in weighing the precious metals, and jewels, and sometimes in philosophical experiments.

Table.

Irregular scale of values.

24 grains, (gr.)	make	1 pennyweight,	marked	pwt.
20 pennyweights,	"	1 ounce,	"	oz.
12 ounces,	"	1 pound,	"	lb.

Unit Equivalents.

lb.		oz.		pwt.		gr.
				1	=	24
		1	=	20	=	480
1	=	12	=	240	=	5760

NOTE 1.—The standard of weight at the United States Mint is the *Troy pound.* It is the same as the Imperial Troy Pound of Great Britain. It is equal to the weight of 22.794422 cubic inches of distilled water, at $39\frac{83}{100}$ degrees, Fahrenheit, the barometer standing at 30 inches.

NOTE 2.—It is not known why this is called *Troy* weight. Some suppose the name to be derived from *Troy Novant,* the old monkish name for London; some, from *Troyes,* a city of France; and some, from *trois,* (French for *three,*) alluding to the three denominations in it, *pwt., oz.,* and *lb.* It is believed that this weight was introduced into Europe from Cairo, in Egypt, in the twelfth century.

NOTE 3.—The name *grain* is supposed to have originated from a very early use, in England, of the weight of an average grain of good wheat as a unit, from which to determine the larger units of weight. The name *pennyweight* probably alludes to the weight of the *silver penny* then in use. All that is now meant by a *grain* is $\frac{1}{5760}$ of a Troy pound. The name *pennyweight* has no peculiar significance now.

NOTE 4.—Precious stones are weighed by *Diamond Weight,* thus:—16 parts = 1 carat grain, (= $\frac{4}{5}$ Troy gr.); 4 carat gr. = 1 carat, (= $3\frac{1}{5}$ Troy gr.)

Care must be taken not to confound the *carat,* which is a definite weight, with the *carat* used in estimating the fineness of gold, which is not a definite weight, but expresses the proportion $\frac{1}{24}$ of the mass referred to.

2. APOTHECARIES' WEIGHT.

Art. 231. Apothecaries' Weight is used in compounding medical prescriptions.

TABLE.

Irregular scale of values.

20 grains, (gr.)	make	1 scruple,	℈ or sc.
3 scruples,	"	1 dram,	ʒ or dr.
8 drams,	"	1 ounce,	℥ or oz.
12 ounces,	"	1 pound,	℔ or lb.

UNIT EQUIVALENTS.

						℈.		gr.
				ʒ.		1	=	20
		℥.		1	=	3	=	60
lb.		1	=	8	=	24	=	480
1	=	12	=	96	=	288	=	5760

NOTE.—In this weight the grain, ounce, and pound are the same as in Troy Weight.

3. AVOIRDUPOIS WEIGHT.

Art. 232. Avoirdupois Weight is used in weighing almost all articles taken in large quantity, such as groceries, drugs, the baser metals, &c., and ordinary commercial transactions.

TABLE.

Irregular scale of values.

16 drams, (dr.)	make	1 ounce,	oz.
16 ounces,	"	1 pound,	lb.
100 pounds,	"	1 hundredweight, or cental,	cwt.
20 hundredweight, or 2000 lbs.	"	1 ton.	

UNIT EQUIVALENTS.

						oz.		dr.
				lb.		1	=	16
		cwt.		1	=	16	=	256
T.		1	=	100	=	1600	=	25600
1	=	20	=	2000	=	32000	=	512000

NOTE 1.—The standard Avoirdupois Pound of the United States is equal to the weight of 27.701554 cubic inches of distilled water at 39.83 degrees of Fahrenheit's thermometer, when the barometer is at 30 inches. It is determined from the Troy pound, the Troy having 5760 grains, and the Avoirdupois having 7000 of the same grains. Hence 1 lb. Av. is 7000 — 5760 = 1240 grs. heavier than 1 lb. Troy.

NOTE 2.—Formerly the *cwt.* was 112 lbs., the *quarter* was 28 lbs., and the *ton* was 20 × 112 lb. = 2240 lb. This ton is sometimes called the *long ton.* It is still in use in England, in the United States custom houses and in a few wholesale trades. There is no denomination in use, called the *quarter*, of 25 lbs. Sixteenths of an ounce, called *drams*, are not in use in common commercial transactions. The customary denominations are *tons*, *pounds*, and *ounces;* and the common fractions of the ounce are *halves*, and *quarters*.

NOTE 3.—The laws of most States define the number of pounds which shall be considered a bushel of various grains, fruits, &c.

TABLE OF AVOIRDUPOIS POUNDS IN A BUSHEL.

	California.	Connecticut.	Illinois.	Indiana.	Iowa.	Kentucky.	Louisiana.	Maine.	Massachusetts.	Michigan.	Minnesota.	Missouri.	New Jersey.	New York.	Ohio.	Oregon.	Pennsylvania.	Vermont.	Wisconsin.
Barley	50	...	48	48	48	48	32	...	46	48	48	48	48	48	48	46	47	46	48
Beans	...	...	60	60	60	60	...	...	...	...	...	60	...	62	...	...	60	...	...
Buckwheat	40	45	40	50	52	52	...	...	46	42	42	52	50	48	...	42	48	46	42
Castor Beans	...	...	46	46	46	...	...	...	...	...	...	46	...	...	...	...	46	...	...
Clover Seed	...	...	60	60	60	60	...	...	...	60	60	60	64	60	60	60	62	...	60
Dried Apples	...	...	24	25	24	...	...	...	...	28	28	24	...	...	...	28	...	...	28
Dried Peaches	...	...	33	33	33	...	...	...	...	28	28	33	...	...	...	28	...	...	28
Flax Seed	...	...	56	56	56	56	...	...	...	...	...	56	55	55	56	...	56	...	56
Hemp Seed	...	...	44	44	44	44	...	...	...	...	.	44	...	...	...	...	...	...	...
Indian Corn	52	56	52	56	56	56	56	...	56	56	56	52	56	58	56	56	56	56	56
In. Corn in ear	...	...	70	68	68	...	...	...	...	...	...	...	...	...	70	...	70	...	...
In. Corn Meal	...	...	48	50	...	50	...	50	50	...	...	...	...	...	...	...	48	...	...
Bitum. Coal	...	...	...	...	...	80	...	...	...	...	...	...	...	...	...	...	76	...	...
AnthraciteCoal	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	*	...	...
Oats	32	28	32	32	35	$33\frac{1}{3}$	32	30	30	32	32	35	30	32	32	34	32	32	32
Onions	...	...	57	48	57	57	...	...	52	...	...	57	...	...	...	...	...	...	...
Potatoes	...	60	60	60	60	60	...	60	...	...	...	60	60	60	...	60	...	60	60
Rye	54	56	54	56	56	56	32	...	56	56	56	56	56	56	56	56	56	56	56
Salt*	...	...	...	50	50	50	...	...	...	...	...	50	...	56	...	...	†	...	...
Timothy Seed	...	...	45	45	45	45	...	...	...	...	...	45	...	44	42	...	45	...	46
Wheat	60	56	60	60	60	60	60	...	60	60	60	60	60	60	60	60	60	60	60

*In Pennsylvania 2240 lbs. make a ton.
† In Pennsylvania 80 lbs. coarse, 70 lbs. ground, or 62 lbs. fine salt make 1 bushel.

4. METRIC MEASURE OF WEIGHT.

Art. 233. The standard of the measure of weight in the Metric System is the *gramme,* (pronounced *gram.*) It is equal to the weight of a cubic centimetre of distilled water, in a vacuum, at 39.2 degrees of Fahrenheit's thermometer.

TABLE.

Regular scale of values, 10.

10 **grammes**, (G.)	equal	1 decagramme,	marked	D.G.
10 decagrammes,	"	1 hectogramme,	"	H.G.
10 hectogrammes,	"	1 kilogramme,	"	K.G.
10 kilogrammes,	"	1 myriagramme,	"	M.G.
$\frac{1}{10}$ **grammes**,	equals	1 decigramme,	"	d.g.
$\frac{1}{10}$ decigramme,	"	1 centigramme,	"	c.g.
$\frac{1}{10}$ *centigramme*,	"	1 milligramme,	"	m.g.

NOTE 1.—The gramme is equal to 15.432 grains; the decagramme 0.3527 Avoirdupois ounces; the hectogramme 3.5274 ounces, and is the weight of a decilitre of water; the kilogramme 2.2046 Avoirdupois pounds, and is the weight of a litre of water; the myriagramme 22.046 Avoirdupois pounds, and is the weight of 10 litres of water.

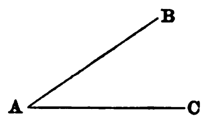
NOTE 2.—The kilogramme is the denomination most used for ordinary transactions. In weighing large quantities the *quintal*, equal to 100 kilogrammes, (plural *quinteaux*,) and the *tonneau*, equal to 1000 kilogrammes, (plural *tonneaux*,) are often employed. The *tonneau* is equal to 2204.6 Avoirdupois pounds, which is 35.4 pounds less than the long ton. It is the weight of a cubic metre of water.

V.

MEASURES OF DIVERGENCE, OR ANGLE.

Art. 234. **Divergence** is difference of direction.

A **plane angle** is the divergence of two straight lines, which meet. It is briefly called an *angle*. Thus, the two straight lines A B and A C, in meeting, are said to form the angle B A C. Such lines are called the *sides of the angle*.



The **vertex** of an angle is the point at which its sides meet. Thus, the vertex of the angle B A C is the point A.

A **circle** is a plane figure bounded by a convex line, called a **circumference**, which is in every part equally distant from a point, called the **centre**, in the same plane.

An **arc** is a part of a circumference.

An angle is measured by making its vertex the centre of a circle, and reckoning the part of the circumference included between its sides. For this reason the measure of angles is often called *circular measure*.

1. ENGLISH AND U. S. CIRCULAR MEASURE.

Art. 235. The common unit of measure for angles or arcs is the **degree**, which is $\frac{1}{360}$ of a circumference.

NOTE.—Since the measure must be the same kind of quantity as the thing measured, a degree is *a unit of divergence*, because it measures divergence. Hence, it is $\frac{1}{360}$ of all the possible divergence about a point in the same plane. Since arcs are proportioned to the angles at their centres, it is customary to call a degree a part of a circumference, instead of a part of the divergence at its centre.

TABLE.

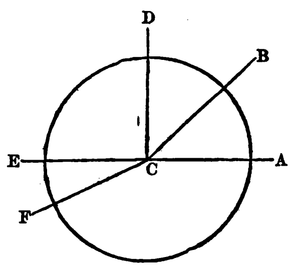
Irregular scale of values.

60 seconds, (″)	make	1 minute,	′
60 minutes,	"	1 degree,	°
360 degrees,	"	1 circumference,	C.

UNIT EQUIVALENTS.

C.		°		′		″
				1	=	60
		1	=	60	=	3600
1	=	360	=	21600	=	1296000

ILLUSTRATION.



1. The lines CE and CD include a quarter of a circumference. Hence their divergence is $\frac{1}{4}$ of 360° = 90°. Such an angle as ECD is called a *right angle.* Two lines forming a right angle are said to be *perpendicular to each other.*

2. The lines CA and CB include one-eighth of a circumference. Hence their divergence is $\frac{1}{8}$ of 360° = 45°. All angles less than 90° are called *acute angles.*

3. The lines CA and CF include more than a quarter of a circumference. Such angles are called *obtuse angles.*

NOTE 1.—Degrees, being 360ths of a circumference, vary in length according to the size of the circle on which they are reckoned.

NOTE 2.—In Astronomy, the circumference of the Zodiac is divided into twelve equal parts, called *signs:* hence 30° make 1 sign.

NOTE 3.—In Geography, *degrees of latitude* are 360ths of the circumference of the Earth, reckoned on a *meridian:* and *degrees of longitude* are 360ths of the circumference of the Earth, reckoned either on the Equator or on a circle parallel to it. The Earth's circumference on a meridian is about 24855 statute miles. This, divided by 360, gives $69\frac{1}{24}$ miles for the average length of a degree of latitude: $\frac{1}{360}$ of the Earth's circumference on the Equator is about $69\frac{1}{7}$ miles.

NOTE 4.—To distinguish the minutes and seconds of this table from those of time, those of circular measure are called *minutes and seconds of arc.* A *nautical,* or *geographical, mile,* is a minute of arc on the Earth, that is, $\frac{1}{60}$ of a degree of latitude.

NOTE 5.—One-fourth of a circumference is sometimes called a *quadrant;* one-sixth, a *sextant;* and one-eighth, an *octant.*

2. DECIMAL CIRCULAR MEASURE.

Art. 236. The French have devised, and used to some extent, a decimal division of the **quadrant.** In this measure, the unit is the **grade,** which is $\frac{1}{100}$ of a quadrant.

TABLE.

Regular scale of values, *centesimal.*

100 seconds,	make	1 minute.
100 minutes,	"	1 grade.
100 grades,	"	1 quadrant.

NOTE 1.—In this measure, since 100 *grades* equal 90 *degrees*, English, 1 grade = .9°, 1 minute = .009°, and 1 seconde = .00009°.

NOTE 2.—In this measure, whichever denomination is taken as the integer, the lower denominations, if any, would be written as a decimal.

VI.

MEASURE OF TIME.

Art. 237. Time is measured duration. To an inhabitant of the Earth, the natural units of time are days and years.

A **day** is the time in which the Earth revolves on its axis.

A **year** is the time in which the Earth moves around the sun.

These natural units have been artificially divided and arranged as in the following

TABLE.

Irregular scale of values.

60 seconds, (sec.)	make	1 minute,	min.
60 minutes,	"	1 hour,	hr.
24 hours,	"	1 day,	da.
7 days,	"	1 week,	wk.
12 calendar months,	"	1 year,	yr.
365 days, (or 52 wk. 1 da.)	"	1 common year.	
366 days, (or 52 wk. 2 da.)	"	1 leap year.	
365 da. 5 hr. 48 min. 47.8 sec.	"	1 solar year.	
100 years,	"	1 century.	

NOTE 1.—The *civil year*, that is, the year recognized in the usages of society, is arranged for civil purposes into weeks and months.

Weeks are arrangements of the days into sevens. In England and the United States the days of the week, beginning with the first, are Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday.

Calendar months are such arrangements of the days as are called the months of the year in a calendar. In England and the United States the calendar months, beginning with the first, are January, February, March, April, May, June, July, August, September, October, November, December.

Seven of the calendar months have 31 days each, namely, January, March, May, July, August, October, and December.

Four of the months have 30 days each, namely, April, June, September and November.

February has 28 days, except in leap-year, when it has 29.

NOTE 2.—The word *day* signifies different quantities of time according to its definition and name. The reasons are that different phenomena are taken as the limits of a day. The Earth turns on its axis from west to east with a uniform motion, and, in so doing, carries the meridian of a place on the Earth past the heavenly bodies. This causes these bodies to appear to move from east to west, and pass anything perpendicular in the plane of the meridian.

An *apparent solar day* at any place is the time between two consecutive passages of the centre of the Sun over the meridian of that place. Such days are not equal, first, because the actual velocity of the Earth in its orbit around the Sun, and, consequently, the apparent daily motion of the Sun eastward around the heavens, are not equal, so that the meridian of a place must in some cases turn farther eastward to overtake the Sun than in others; and, secondly, because of the inclination of the Earth's axis to the plane of its orbit.

A *mean solar day* is the average length of all the solar days of the year. Of course, mean solar days are equal.

Apparent time is time measured by apparent solar days.

Mean time is time measured by mean solar days. Nearly all the time recognized in the usages of society is mean time.

A *sidereal day* at any place is the time between two consecutive passages of a fixed star over the meridian of that place. It is the time of a complete revolution of the Earth on its axis. The solar day is nearly four minutes longer than the sidereal day, (3 min. 55.9 sec. of mean solar time.)

NOTE 3.—The civil day of any place, that is, the day recognized in the usages of society, commences and ends at midnight of that place, that is, when the Sun is on that part of the meridian of that place which is on the opposite side of the Earth. It is usually reckoned on clock dials in two parts of 12 hours each, namely, *ante meridian*, (A. M.,) or forenoon from midnight to noon, and *post meridian*, (P. M.,) or afternoon from noon to midnight.

The astronomical day of any place, that is, the day used in astronomical calculations, commences at noon, and extends 24 hours to the next noon. In such time, 18 hours, or 18 o'clock, is 6 o'clock in the morning of the next civil day, &c.

NOTE 4.—The word *year* signifies different quantities of time accord-

ing to its definition and name. The reasons are that different phenomena are taken as the limits of a year. The Earth moves around the Sun from west to east, causing the Sun to appear to move through and around the starry firmament, from west to east, in a certain number of the Earth's days.

A *sidereal year* is the time intervening between two consecutive apparent passages of the Sun's centre by a fixed star. It is the time of the Earth's complete circuit of its orbit.

A *common*, or *tropical*, year, sometimes also called the *Equinoctial year*, is the time intervening between two consecutive passages of the Sun's centre across the Equator at the vernal Equinox. Astronomers have stated slightly different values for the length of the Equinoctial year, but it is generally taken by them as 365.24222 *mean solar days*, equal to 365 da. 5 hr. 48 min. 47.808 sec. The length of the sidereal year is taken by astronomers at 365.256374416 mean solar days, equal to 365 da. 6 hr. 9 min. 10.7 sec. mean solar time. (Chauvenet.)

The reason that the Equinoctial year is shorter than the sidereal is that the vernal Equinox moves annually westward about 50″.223, so that the Sun, in its apparent motion eastward around the heavens, comes to that point before it has made an entire circuit.

NOTE 5.—An *era* is a reckoning of years from some event. The Christian Era is reckoned from the supposed date of the birth of Christ. The years before that event are by Christians styled B. C., meaning *before Christ*. The years of the Christian Era are styled A. D., meaning *Anno Domini*, Latin for *in the year of our Lord*.

NOTE 6.—The Christian *civil year* begins and ends at midnight between December 31 and January 1. It varies in length according to the following

RULE.—*Centennial years which are divisible by 400 without a remainder, and such other years as are divisible by 4 without a remainder, contain 366 days. Other years contain 365 days.*

By this rule, the years 1600, 2000, 2400, &c., A. D., must contain 366 days: also such years as 1848, 1852, 1856, 1860, 1864, and 1868.

A year of 366 days is called *bissextile*, or *leap-year*. The reason for the foregoing rule may be thus stated:—A civil year of 365 days falls short of a true tropical year by 5 hr. 48 min. 47.8 sec. In four years the civil year is 4 times as much behind the true date, namely, 23 hr. 15 min. 11.2 sec., which is nearly a day. Hence by the rule a whole day is added to the civil year once in four years. But this is adding too much by 44 min. 48.8 sec., and would in 100 years put the civil year ahead of the true date by 25 times 44 min. 48.8 sec., namely, 18 hr. 40 min. 20 sec., which is over three-fourths of a day. Therefore, reckoning the centennial year only 365 days (although divisible by 4) puts the civil year (24 hr. — 18 hr. 40 min. 40 sec. =) 5 hr. 19 min. 20 sec. behind the true date. This is nearly a quarter of a day. Therefore, reckoning the fourth centennial year (those divisible by 400) as 366 days puts the civil year (24 hr. — 4 times 5 hr. 19 min. 20 sec. =) 2 hr. 42 min. 40 sec. ahead of the true date. This would be a day too much in about 3541 years.

NOTE 7.—Julius Cæsar, 46 B. C., decreed that three consecutive years should have 365 days each, and the fourth year 366 days, thus making the civil year average 365¼ days. He placed the additional day in the

fourth year between Feb. 24 and Feb. 25. Since Feb. 25 was, in the Roman calendar, *Sexto Calendas Martii*, that is, *the sixth of the Calends of March*, this additional day was called *Bis Sexto Calendas Martii*, and the year which contained it was called *Bissextile Year*. Among moderns this year is generally called *leap-year*, and the additional day is Feb. 29.

NOTE 8.—The average civil year, as established by Julius Cæsar, is called the Julian Year. It reckoned too much time for a year, so that, in A. D. 1582, the dates of the civil year were 10 days behind those of the solar year. To rectify this, Pope Gregory XIII. decreed that Oct. 5, 1582, should be reckoned October 15. He also abolished the rule established by Cæsar, and instituted the rule now in force, given in Note 6. This amended calendar is called the Gregorian Calendar, and time reckoned by it is called New Style, (N. S.) Time reckoned by the Julian Calendar is called Old Style, (O. S.)

NOTE 9.—As A. D. 1600 was a leap-year in both styles, the difference between New Style and Old Style was ten days in the interval between 1582 and 1700. But 1700 was a Julian leap-year, but a Gregorian common year. Hence, from 1700 to 1800, Old Style dates were 11 days behind New Style. For the same reason Old Style dates, from 1800 to 1900, are 12 days behind New Style, and from 1900 to 2100 will be 13 days behind.

NOTE 10.—New Style and the Gregorian rule were adopted by Italy, Spain, Portugal, France, and the Catholic States of Germany, in A. D. 1582; by Scotland in A. D. 1600; by Denmark, Sweden, and the Protestant States of Germany about A. D. 1700; and by Great Britain A. D. 1752. The British Parliament decreed that the day following Sept. 2, 1752, should be reckoned Sept. 14, because Old Style dates were then 11 days behind New Style. Russia and Greece still use Old Style and the Julian reckoning, and, hence, Russian and Greek dates are now 12 days behind ours.

NOTE 11.—Dates are referred to different eras by different peoples. Most Christians reckon dates from the beginning of the Christian Era. Jews sometimes reckon their dates from the supposed date of the creation of the world. This is called the Jewish Era. The year 5630 of the Jewish Era commenced Sept. 6, 1869 of the Christian Era. The Mohammedans reckoned from the date of the Hegira, or flight of Mohammed from Mecca to Medina, namely, A. D. July 16, 622. The year 1286 of the Mohammedan Era commenced A. D. April 13, 1869. Jewish and Mohammedan years are variable in length, being reckoned by the motions of the Moon, instead of those of the Sun. There are several other chronological eras in use among the different nations of the Earth. Some of these, referred to in history, are the Era of the Foundation of Rome, of Nabonassar, of the Olympiads, of the Seleucidæ, of Diocletian, the Byzantine Era, and the Julian Period. For a description of these, the learner may consult the encyclopedias.

NOTE 12.—To interpret dates in the Christian Era, consider that the year A. D. 1 began at the birth of Christ. Then at 12 o'clock at noon May 20, 1871, the time considered as having actually elapsed is 1870 years, 4 months, 19 days, 12 hours, since the beginning of the era; or, *ordinally*, it is the 12th hr. of the 20th day of the 5th month of the *1871st* year.

VII.

Measures of Trade Value.

Art. 238. Money is that which is used in trade as a representative of value. The money of a community, state, or nation, is frequently called its *currency*, because it is *current*, that is, passes readily from one person to another in trade.

In most nations the materials used for money are the metals gold, silver, and copper, or their alloys. Many commercial nations also use, for convenience, printed and written *promises* to pay a certain sum of money. These are called *notes*, or *bills*, and, when used as money, are called *paper money*, or *paper currency*.

The material of which money is made is frequently called the *circulating medium*.

Art. 239. The metallic money of a country is usually made and issued by the government of that country.

A **coin** is a piece of metal used as money, bearing marks impressed upon it to designate its value.

Bullion is uncoined gold or silver. It is usually in bars or pieces called *ingots*.

Art. 240. An **alloy** is a compound produced by melting together two or more metals.

The purity of gold is estimated by *twenty-fourths*, called **carats**. Pure gold is 24 carats fine, because $\frac{24}{24}$, or the whole of it, is gold. A piece, of which $\frac{22}{24}$ is gold, is said to be *22 carats fine*, &c.

Coins are usually alloys, because pure gold and silver are too soft to be as durable as coin should be, or because the coin can be made more convenient in size, &c., to represent its value.

Art. 241. A **legal tender** is that money which, when offered as payment for a debt, or obligation, is in law a satisfaction of that debt or obligation.

NOTE.—The Constitution of the United States forbids a State to make anything but gold and silver a legal tender in payment of debt, but Congress has sometimes made the notes of the United States a legal tender.

Art. 242. The **money of account** of any state or nation is money expressed in those denominations in which business is transacted in that state or nation. It is sometimes different from the money in circulation.

1. UNITED STATES MONEY.

Art. 243. United States Money, sometimes called *Federal Money,* is the legal currency of the United States. It was adopted by Congress August 8, 1786.

TABLE.

Regular scale of values.

10 mills, (marked m.)	make	1 cent,	marked	ct.
10 cents,	"	1 dime,	"	d.
10 dimes,	"	1 dollar,	"	$.
10 dollars,	"	1 eagle,	"	E.

Art. 244. The coins of the United States are made of alloys of gold, silver, nickel, and copper. The *mill* is not coined.

1. **Of copper,** or *alloyed copper:*—The *cent,* and *two-cent piece.*

2. **Of nickel:**—The *three-cent piece* and *five-cent piece.*

3. **Of alloyed silver:**—The *three-cent piece, half dime, dime, quarter-dollar, half-dollar,* and *dollar.*

4. **Of alloyed gold:**—The *dollar, quarter-eagle, three-dollar piece, half-eagle, eagle, double eagle,* and *fifty-dollar piece.*

Art. 245. The gold coins of the United States are of an alloy of $\frac{9}{10}$ pure gold and $\frac{1}{10}$ an alloy, consisting of equal parts of silver and copper. It is 21.6 carats fine.

The silver coins are $\frac{9}{10}$ pure silver and $\frac{1}{10}$ copper, except the three-cent piece, which is 3 parts silver, and 1 part copper.

2. Canada Money.

Art. 246. In A. D. 1858, the Canadian Parliament adopted a decimal currency, whose denominations are the same as these of U. S. money. Previous to 1858, Canadian money had the same denominations as English money.

The coins of Canada money are made of (silver, and copper:)

Of silver:—The shilling or 20-cent piece, the 10-cent piece, and 5-cent piece.

Of copper:—The cent.

Art. 247. The silver coins consist of 925 parts (.925) pure silver and 75 parts (.075) copper.

NOTE.—The value of the 20-cent piece in the United States is $18\frac{2}{3}$ cents, of the 10-cent piece $9\frac{1}{3}$ cents, and of the 5-cent piece $4\frac{2}{3}$ cents.

3. English Money.

Art. 248. English Money, sometimes called **Sterling,** is the legal currency of Great Britain.

Table.

Irregular scale of values.

4 farthings, (far. or qr.)	make	1 penny,	marked	d.
12 pence,	"	1 shilling,	"	s.
20 shillings,	"	1 pound,	"	£.

Unit Equivalents.

				d.		far.
		s.		1	=	4
£.		1	=	12	=	48
1	=	20	=	240	=	960

English gold coins are $\frac{11}{12}$ pure gold, and $\frac{1}{12}$ alloy, that is, 22 carats fine. They are the *sovereign* (£1) and *half-sovereign.*

English silver coins are $\frac{925}{1000}$ pure silver, and $\frac{75}{1000}$ copper. They are the *crown,* (£$\frac{1}{4}$), *half-crown,* (£$\frac{1}{8}$), *shilling,* and *sixpence,* ($\frac{1}{2}$ s.)

English copper coins are the *penny, half-penny,* and *farthing.*

NOTE 1.—A gold coin of 21 shillings value, called a *guinea,* and one of 10 shillings 6 pence, called a half-guinea, were once made.

NOTE 2.—A pound is equal to $4.86 U. S. money.

4. FRENCH MONEY.

Art. 249. French Money, the legal currency of France, is in denominations of a uniformly decimal scale.

TABLE.

10 centimes,	make	1 decime.
10 decimes,	"	1 franc.

French gold coins are $\frac{9}{10}$ gold, and $\frac{1}{10}$ copper, that is, 21.6 carats fine. They are pieces of 100, 50, 20, 10, and 5 francs.

French silver coins are $\frac{9}{10}$ silver and $\frac{1}{10}$ copper. They are pieces of 5, 2, 1, $\frac{1}{2}$, and $\frac{1}{5}$ francs. The franc weighs 5 grammes.

French copper, or bronze, coins are $\frac{95}{100}$ copper, $\frac{4}{100}$ tin, and $\frac{1}{100}$ zinc. They are pieces of 10, 5, 2, and 1 centimes.

French *money of account* is in francs and centimes, thus,—100 centimes make 1 franc.

NOTE.—A franc is equal to 18$\frac{3}{5}$ cents U. S. money.

Art. 250. The moneys of account and the coins of the various parts of the world are so numerous that bankers and traders use extensive tables of them. These tables are amended from time to time, to make them correspond to the changes which take place in money. The tables are too extensive to insert in a school text-book.

Miscellaneous Tables.

1. Counting.

12 units,	make	1 dozen.	12 gr., or 1728,	make	1 gt. gr.
12 dozen, or 144	"	1 gross.	20 units	"	1 score.

2. Paper.

24 sheets,	make	1 quire.	2 reams,	make	1 bundle.
20 qrs., or 480 s.,	"	1 ream.	5 bundles,	"	1 bale.

3. Books.

A folio book,	(fol.)	is made	of sheets	folded in	2	leaves.
A quarto,	(4to.)	"	"	"	4	"
An octavo,	(8vo.)	"	"	"	8	"
A duodecimo,	(12mo.)	"	"	"	12	"
An 18mo.,		"	"	"	18	"
A 24mo.,		"	"	"	24	"

4. Weights.

196 pounds of flour,	make	1 barrel.
200 pounds of beef, pork, or fish,	"	1 barrel.
560 pounds wheat, (8 bu. 70 lbs. each,) in England,	"	1 quarter.
7 pounds, common articles, in England,	"	1 clove.
14 pounds, " " " "	"	1 stone.

5. Metric Measures of Length.

Names.	Value in Metres.	Equivalents.
Myriametre	10000.	6.2137 miles.
Kilometre	1000.	0.62137 mi., or 3280 feet 10 inches.
Hectometre	100.	328 feet 1 inch.
Decametre	10.	393.7 inches.
Metre	1.	39.37 inches.
Decimetre	.1	3.937 inches.
Centimetre	.01	.3937 inch.
Millimetre	.001	.0394 inch.

6. Metric Measures of Surface.

Names.	Value in Square Metres.	Equivalents.
Square Myriametre..........	100000000	24711.43 acres.
Square Kilometre............	1000000	247.1 acres.
Square Hectometre, or *Hectare*	10000	2.471 acres.
Square Decametre, or Are	100.	119.6 square yards.
Square Metre, or *Centare*	1.	1550 square inches.
Square Decimetre.............	.01	15.5 square inches.
Square Centimetre............	.0001	.155 square inch.
Square Millimetre............	.000001	.00155 square inch.

7. Metric Measures of Capacity.

Names.	Litres.	Cubic Measure.	Dry Measure Equivalents.	Liquid Measure Equivalents.
Kilolitre, or Stere ..	1000.	1 cubic metre	1.308 cu. yd....	264.17 gallons.
Hectolitre ...	100.	$\frac{1}{10}$ cubic metre......	2 bu. 3.35 pk...	26.417 gallons.
Decalitre	10.	10 cubic decimetres	9.08 quarts......	2.6417 gallons.
Litre	1.	1 cubic decimetre.	0.908 quarts.....	1.0567 quarts.
Decilitre......	.1	$\frac{1}{10}$ cubic decimetre.	6.1022 cu. in....	0.845 gills.
Centilitre.....	.01	10 cubic centimetres	0.6102 cu. in....	0.338 fluidounce.
Millilitre.....	.001	1 cubic centimetre	0.061 cu. in......	0.27 fluidrachm.
Cubic Millimetre	.000001	....	0.000061 cu. in.	

8. Metric Measures of Weight.

Names.	Grammes.	Quantity of Water.	Equivalents.
Tonneau	1000000.	1 cubic metre......	2204.6 lb. Av.
Quintal.......................	100000.	1 hectolitre..........	220.46 lb. Av.
Myriagramme.............	10000.	10 litres...............	22.046 lb. Av.
Kilogramme...............	1000.	1 litre	2.2046 lb. Av.
Hectogramme.............	100.	1 decilitre...........	3.5274 oz. Av.
Decagramme...............	10.	10 c. m.^{3}	0 3527 oz. Av.
Gramme......................	1.	1 c. m.^{3}	15.432 grains.
Decigramme...............	.1	$\frac{1}{10}$ c. m.^{3}	1.5432 grains.
Centigramme	.01	10 m. m.^{3}...............	0.1543 grains
Milligramme...............	.001	1 m. m.^{3}...............	0.0154 grains.

SYNOPSIS OF DENOMINATE NUMBERS.

Kinds of Denominate Quantities.

- **Length.**
  - English.
    - Common.
    - Cloth.
    - Duodecimal.
    - Surveyors'.
    - Miscellaneous.
  - Metric.
- **Surface.**
  - English.
    - Common.
    - Surveyors'.
  - Metric.
- **Volume.**
  - English.
    - Common.
    - Capacity.
      - English.
        - Liquid.
          - Common.
          - Apothecaries'.
        - Dry.
      - Metric.
  - Metric.
    - Common.
    - Capacity.
- **Force.**
  - English.
    - Troy Weight.
    - Apothecaries'.
    - Avoirdupois.
  - Metric.
- **Angle.**
  - English.
  - Decimal.
- **Time.**
  - Natural Units.
    - Days.
      - Apparent Solar.
      - Mean Solar.
      - Sidereal.
      - Civil.
      - Astronomical.
    - Years.
      - Sidereal.
      - Tropical.
      - Civil.
        - Common.
        - Leap.
  - Artificial Units.
    - Seconds.
    - Minutes.
    - Hours.
    - Weeks.
    - Months.
    - Centuries.
    - Eras.
      - Christian.
      - Jewish.
      - Mohammedan, &c.
- **Money.**
  - United States.
  - Canada.
  - English.
  - French, &c.

CHAPTER XII.

UNITED STATES MONEY.

OPERATIONS IN UNITED STATES MONEY.

Art. 251. The **money of account** of the United States is in dollars, cents, and mills. The *dollar* is the unit, and cents and mills are expressed as fractions of it.

TABLE.

10 mills,	make	1 cent,	ct.
100 cents,	"	1 dollar.	$.

NOTATION OF UNITED STATES MONEY.

Art. 252. To write numbers in United States money.

Because United States money has a decimal scale, quantities in this currency can be expressed either as pure or mixed decimals, or as regular integers. Since the *dollar* is the unit, the dollars of a sum are its integral part, the *cents* must occupy the tenths' and hundredths' places, and the mills the thousandths' place.

Rule.—*First write the dollars as an integer, then the cents as the first two figures of a decimal at the right, and the mills as the third. Place the dollar sign at the left of the whole.*

NOTE 1.—When cents are not given, their places must be filled with ciphers. When a whole number of cents less than ten is given, the first decimal place must be filled with a cipher. Thus, *five dollars three mills* is written $5.003; *five dollars seventy-five cents three mills* is $5.753; *two dollars seven cents five mills* is $2.075. *Sixty-two cents five mills* may be written $0.625, or $.625, but it is advisable to write a cipher in the dollar's place when the sum is less than a dollar.

NOTE 2.—Cents are often written as a common fraction. Thus, $2.25 may be written $2\frac{25}{100}$. This may be read *two dollars and twenty-five hundredths*, or *two and twenty-five hundredths dollars*.

NOTE 3.—Common fractions of a cent may be written as such, or as mills when possible. Thus, *one dollar thirty-seven and a half cents* may be written \$1.37½, or \$1.375.

EXAMPLES FOR PRACTICE.

1. Write two dollars, sixty-two cents, five mills.
2. Write ten dollars, thirty cents, three mills.
3. Write twenty dollars, six cents, two mills.
4. Write thirty dollars, five cents, one mill.
5. Write one hundred five dollars, fifty cents.
6. Write two hundred dollars, two cents, two mills.
7. Write three thousand two dollars, three cents.
8. Write sixty dollars, seven and three-tenths mills.
9. Write four thousand forty dollars, four cents, five and five-tenths mills.

NUMERATION OF UNITED STATES MONEY.

Art. 253. To read numbers in United States money, when expressed decimally.

Rule.—*Read all on the left of the separatrix as dollars, the next two figures as cents, the next figure as mills, and any remaining figures as a decimal of a mill.*

ILLUSTRATION.—The number \$1.3125 may be read *one dollar thirty-one cents two mills and five-tenths of a mill*, or *one dollar thirty-one cents and a quarter*, &c. The number \$9.87575 may be read *nine dollars eighty-seven cents five mills and seventy-five hundredths of a mill.*

EXAMPLES FOR PRACTICE.

Read the following numbers:—

1. \$4.103.	7. \$24.62½.	13. \$100.253.
2. \$8.012.	8. \$35.00.	14. \$1000.5625.
3. \$10.30.	9. \$43.06.	15. \$604.4375.
4. \$13.54.	10. \$20.005.	16. \$500.5004.
5. \$30.303.	11. \$85.045.	17. \$404.4044.
6. \$31.136.	12. \$105.105.	18. \$6006.6666.

Art. 254. Since in United States money the dollars are integers, and cents and mills are decimals of a dollar,

addition, subtraction, multiplication and division may be performed in the same manner as in other decimals.

Examples for Practice.

1. A man paid \$11.25 for pork; \$10.875 for flour; \$5.625 for butter; and \$6.35 for fish: how much did he pay for all? Ans. \$34.10.

2. A grocer bought coffee for \$65.375; sugar for \$80.50; tea for \$44.125; potatoes for \$20.65; and apples for \$13.75: what was the cost of the whole?

3. A man having \$4628.125, gave \$3296.50 for a farm; how much money has he remaining? Ans. \$1331.625.

4. A's farm cost him \$6827.75, and he sold it for \$8112.50; how much did he gain?

5. What cost 43 bushels of wheat at \$1.625 per bushel?

6. What cost 17 oranges at \$0.08 apiece?

7. How many hats at \$2.25 each can be bought for \$29.25?

8. How many bushels of onions at \$1.125 per bushel can be bought for \$18.? Ans. 16.

9. At \$.23 per pound, what costs 15 pounds of butter?

10. How many pounds of nails at 8 cents per pound can be bought for \$2? Ans. 25.

11. James paid \$.87½ for a reader, \$.75 for an arithmetic, \$.25 for a slate, \$1. for a grammar, and \$.125 for paper; what did he pay for all?

12. How many bushels of apples can be bought for \$58.1875, at \$.43¾ per bushel? Ans. 133.

13. A has \$5271.3125, and B has \$4627.87½; how much more has A than B?

14. If 1 man earn \$1.625 in a day, how much can 15 men earn in the same time?

15. At \$7.50 per barrel, how many barrels of flour can be bought for \$60?

16. If 35 bushels of corn cost \$22.05, how much will 1 bushel cost?

17. What will 1 gallon of molasses cost, if 33 gallons cost \$31.35?

18. How many pounds of fish, at \$.08 per pound, can be bought for \$5.84?

19. A man receives \$66.30 for 1 month's labor; how much is that a day, allowing 26 working days to the month?

Ans. \$2.55.

20. How many pounds of grapes, at \$.08 a pound, will pay for 18 pounds of raisins, at \$.20 a pound? Ans. 45.

21. A has \$150.25, B has \$12.375 more than A, and C has as much as A and B, less \$15.125; how much has C, and how much have they all? Ans. C, \$297.75; all, \$610.625.

22. Bought an equal number of sheep and hogs for \$2094.75; gave for the sheep \$6.25 each, and for the hogs \$9.50 each; how many of each did I buy? Ans. 133.

23. Gave \$137.75 for a horse, \$135.37½ for a wagon, and \$28.625 for harness, and sold the whole for \$350; how much did I gain? Ans. \$48.25.

24. What is the value of 16 barrels of potatoes, each barrel containing $2\frac{3}{4}$ bushels, at \$.87½ per bushel?

25. A man sold 83 fleeces of wool, averaging 3½ pounds each, at \$.56¼ per pound; what did he receive for his wool?

Ans. \$163.40$\frac{5}{8}$.

26. A has \$27.23, which lacks \$19.25 of being 4 times B's money; and C is worth 4 times as much as A and B together, less \$98; how much money has B and C each?

Ans. B \$11.62, C \$57.40.

27. A drover had 150 sheep, which he might have sold for \$7.50 a head, and gained \$51, but after holding, he sold at a loss of \$136.50; how much a head did they cost him, and for how much a head did he sell them?

Ans. cost, \$7.16; sold for, \$6.25.

28. A farmer bought a piece of land containing 187½ acres, at \$36.25 per acre, and sold $\frac{1}{3}$ of it at a profit of \$578$\frac{1}{8}$; at what price per acre was the land sold? Ans. \$45.50.

29. A stock dealer bought 70 horses for \$5950. Seven of them died, and he sold the remainder so as to gain \$98; for how much did he sell them each? Ans. \$96.

Art. 255. In the determination of cost from price, there are two cases.

Case I.

Art. 256. To find the cost of two or more articles from the price of one of them.

Ex. 1. What cost 7 hats at \$4.37 5 apiece? Ans. \$30.62 5.

Written Process.	Explanation.
\$4.37 5	At \$4.37 5 apiece, 7 hats cost 7 times \$4.37 5, that is, \$30.62 5.
7	
\$30.62 5	Note.—In multiplying, we may either consider the cents and mills as denominations, or as a decimal of the unit dollar.

Rule.—*Multiply the price of one article by the number of articles.*

Examples for Practice.

2. What cost 8 barrels flour at \$6.50 per barrel?
3. At \$15.125 an acre, what will 27 acres of land cost?
4. What will be the cost of 73 bushels of potatoes at 95 cents per bushel?
5. At \$13.75 per ton, what will $15\frac{3}{4}$ tons of hay cost?
6. At \$.03 a quart, what will 141 quarts of milk cost?
7. Bought $2\frac{1}{3}$ pieces of carpet containing 22.5 yards each; what was the whole cost?
8. What will $\frac{3}{7}$ of a yard of ribbon cost at \$.33 a yard?
9. At $\frac{3}{8}$ of a cent each what will 11 peaches cost?
10. At \$$24\frac{5}{8}$ an acre what will $72\frac{7}{9}$ acres of land cost?
11. At \$2.50 a yard what will $\frac{4}{5}$ of a yard of cloth cost?
12. What will $\frac{3}{7}$ of a pound of oats cost at \$.00$\frac{8}{9}$ a pound?

Case II.

Art. 257. To find the cost of a number of articles from the price of a collection of them.

Ex. 1 What cost 198 pencils at $62\frac{1}{2}$ cents per dozen?

Ans. \$$10.31\frac{1}{4}$.

FIRST METHOD.	SECOND METHOD.
$12)\underline{198}$	$12)\underline{\$0.625}$
$16\frac{1}{2}$ dozen.	$\$0.05\frac{5}{24}$
$\$0.625 \times 16\frac{1}{2} = \$10.3125.$	$\$0.05\frac{5}{24} \times 19.8 = \$10.31\frac{1}{4}$

ANALYSIS FIRST.—If 1 dozen pencils cost $62\frac{1}{2}$ cents, 198 pencils, which are $16\frac{1}{2}$ dozen, cost $16\frac{1}{2}$ times $62\frac{1}{2}$ cents, that is, \$10.3125.

ANALYSIS SECOND.—If 1 dozen, that is, 12 pencils, cost $62\frac{1}{2}$ cents, 1 pencil costs $\frac{1}{12}$ of $62\frac{1}{2}$ cents, that is, $5\frac{5}{24}$ cents, and 198 pencils cost 198 times $5\frac{5}{24}$ cents, that is, $\$10.31\frac{1}{4}$ cents.

Rules—I. *Multiply the price of one collection of articles by the number of such collections in the given number.* Or,

II. *Divide the price of the collection by the number of articles in the collection, and multiply the quotient by the whole number of articles.*

NOTE.—The sign C, abbreviated from the Latin *centum*, is often written in business for the word *hundred*, and M, from the Latin *mille*, for *thousand*.

EXAMPLES FOR PRACTICE.

2. What cost 42 lemons at 20 cts per dozen?

3. What will be the cost of 9850 shingles at $\$4.62\frac{1}{2}$ per M.?

4. Required the cost of 15625 pounds of hay at \$12.50 per ton (2000 pounds)? Ans. $\$97.65\frac{5}{8}$.

5. What will be the cost of 13850 feet of flooring boards at \$32 per thousand? Ans. \$443.20.

6. What will be the cost of 160 sacks of guano, each sack containing $156\frac{1}{4}$ pounds, at \$15.50 per ton? Ans. \$84.94.

7. What will be the cost of 750 palings at \$7.50 per M?

8. What cost 132 clothes-pins at \$.08 per dozen? Ans. \$.88.

9. What will be the cost of 186723 pounds of iron at $\$115\frac{1}{2}$ per ton?

10. What cost 23850 bricks at the rate of \$11.50 per thousand?

11. What cost 13620 pounds of wheat at $\$1.87\frac{1}{2}$ per bushel of 60 pounds?

12. How much is the freight on 4260 pounds from Pittsburgh to Altoona, at $\$\frac{7}{8}$ per 100 pounds?

13. What will be the cost of 583 feet of boards at $\$23\frac{1}{2}$ per M; 864 palings at $\$1.37\frac{1}{2}$ per C; 43 pounds of nails at \$8.50 per C, and 1572 bricks at $\$7\frac{5}{8}$ per M? Ans. \$41.222.

ACCOUNTS AND BILLS.

Art. 258. An **account** is a formal statement, made by one party, of his commercial transactions with another.

A **party,** in business, is either one person, or more than one, involved as debtor or creditor in a business transaction.

A **debtor** is a party that owes.

A **debt** is that which is owed.

A **creditor** is a party who has a claim upon another, or to whom a debt is owed.

A **debit** is an entry, made in an account, of the indebtedness of the other party to the maker of the account.

A **credit** is an entry, made in an account, of a claim of the other party upon the maker of the account.

Art. 259. An account may exhibit debits alone, or credits alone, or both together. The maker, or keeper of the account heads it so as to signify that the other party is *in account with him.* Thus, if you keep an account of your transactions with Henry Mann, you head it thus:—

Dr. *Henry Mann.* *Cr.*

The debit side of an account is marked *Dr.* for *debtor,* usually at the left: the credit side is marked *Cr.* for *creditor,* usually at the right. The name of the party debited and credited is written between these abbreviations.

Art. 260. An account is said to be **balanced** when the sum of the credits equals that of the debits.

The **balance** of an account is the quantity which, if added to the less side, would make its sum equal to that of the other side. It may be found by subtracting the sum of the less side from that of the greater.

An account is usually called **current** when it remains open for the entry of transactions as they take place.

Art. 261. A **bill** is, in business, a paper containing a statement of the items of indebtedness of one party to another. It is usually presented by the creditor to the debtor for payment.

A **bill of sale** is a paper containing a statement of things bought by one party from another. It is usually given by the seller to the buyer as evidence of the transaction. When used as evidence of payment it is called a **voucher.**

A bill is said to be **footed** when the amount of the items is written at the foot of the column.

A bill is **receipted** when the creditor, or his agent, signs upon the bill a statement that payment has been received. It is then called a **receipt,** and may be used as a voucher.

A **due-bill** is a written acknowledgment by one party that a certain sum is due from him to another party. The following is a specimen of a simple one, without conditions:—

Due James Lane ten dollars, for value received.

George Merrick.

Pittsburgh, May 4, 1871.

Art. 262. In accounts and bills the symbol @ is often used for the word *at*, before the statement of a price; for example, "*6 chairs @ $4.50.*"

In market quotations @ is often used for the word *to*, before the last quoted price, and the symbol ℔ is often used for the Latin word *per*, meaning *by the;* for example: "*Flour $5.50 @ $6. ℔ bbl.*"

EXAMPLES FOR PRACTICE.

Find the footings and balances in the following bills and accounts:—

(1.) *Pittsburgh, May 25, 1873.*

Mr. G. J. LUCKEY,

Bought of A. H. ENGLISH & Co.

40 Elementary Arithmetics,	@ $ .35		
25 English Grammars,	" .80		
18 Elementary Algebras,	" 1.25		
15 Readers,	" 1.375		
			$77.125.

Received Payment,

A. H. ENGLISH & Co.

(2.) *Pittsburgh, June 15, 1873.*

Mr. H. CRAWFORD,

Bought of MYERS & STEVENSON.

21 Hams,	*2.67 lbs.*	@ $ .14½	
12 Pieces Bacon,	*73 lbs.*	" .12½	
5 " Dried Beef,	*31 lbs.*	" .17	
9 Com. Shoulders,	*144 lbs.*	" .07¾	
1 Tierce Lard,	*347½ lbs.*	" .09¾	
6 Buckets "	*123 lbs.*	" .10½	
1 Case 6 lb. Pails Lard,	*60 lbs.*	" .13¼	

Rec'd Payment, by check on 1st. Nat. Bank,

MYERS & STEVENSON.

(3.) *New York, July 3, 1873.*

Messrs. HALL & BROWN,

Bo't of JONES, SMITH & Co.

135	*lbs.*	*Java Coffee,*	@ $ .28	
210	"	*Rio* "	" $.23\frac{1}{2}$	
324	"	*C. Sugar,*	" $.13\frac{3}{4}$	
106	"	*Rice,*	" $.08\frac{1}{4}$	
32.5	"	*Tea,*	" $1.07\frac{1}{2}$	
36	"	*Cheese,*	" $.08\frac{2}{3}$	

Rec'd Payment, by note at 3 months,

JONES, SMITH & Co.

Per JOHN DALZELL.

(4.) ACCOUNT CURRENT.

Philadelphia, June 18, 1873.

Mr. HENRY SNOW & Co.

To BROOKS, BLACK & Co. *Dr.*

Feb.	*5.*	To *10 tons Bar Iron,*	@	$123.18	$
Mar.	*10.*	" *3* " *Cast Steel,*	"	524.15	
Apr.	*12.*	" *5* " *Rod Iron,*	"	137.50	
"	*24.*	" *10 kegs Nails,*	"	7.50	
May	*2.*	" *3 cwt. Sheet Iron,*	"	6.75	
"	*9.*	" *350 lbs. Spikes,*	"	$.05\frac{1}{2}$	
					$

Cr.

Mar.	*12.*	By *2000 bu. Shelled Corn,*	@	$ $.82\frac{3}{4}$	
"	*25.*	" *112 bbl. XX Flour,*	"	8.75	
Apr.	*8.*	" *558 lbs. Bran,*	"	$.01\frac{1}{3}$	
"	*13.*	" *87* " *Crushed Sugar*	"	$.16\frac{2}{3}$	
May	*2.*	" *Draft on Boston,*			$500
"	*6.*	" *63 lbs. Java Coffee,*	"	.31	
June	*8.*	" *5 bbl. Molasses,*	"	20.50	
					$

Bal. due BROOKS, BLACK & Co. $327.28.

Art. 263. Many of the instances in which money is divided are cases of finding the price of an article from the cost of a known quantity, and of finding the number of articles from their known price and cost.

Case I.

Art. 264. To find the price of a single article, or collection of articles, from their number and cost.

Ex. 1. If 7 pounds of sugar cost \$1, what is the price per pound? Ans. \$0.14$\frac{2}{7}$.

Analysis.—If 7 pounds of sugar cost \$1.00, 1 pound costs $\frac{1}{7}$ of \$1, which is $\frac{1}{7}$ of 100 cents, or 14$\frac{2}{7}$ cents.

2. If 12 dozen of clothes-pins cost \$0.60, what is the price per dozen? Ans. \$0.05.

Analysis.—If 12 dozen of clothes-pins cost 60 cents, 1 dozen costs $\frac{1}{12}$ of 60 cents, which is 5 cents.

Rule.—*Divide the cost by the number of articles, or collections of articles.*

Examples for Practice.

3. If 73 acres of land cost \$5328, how much is that per acre?

4. A man purchased 162$\frac{3}{4}$ tons of hay for \$5330.0625; how much did it cost per ton? Ans. \$32.75.

5. A miller bought 43$\frac{3}{4}$ bushels of wheat for \$70; how much did he pay a bushel? Ans. \$1.60.

6. If the board of a family be \$319.37$\frac{1}{2}$ for 1 year, how much is it per day?

7. If .7 of a ton of iron cost \$87$\frac{1}{2}$, what is the cost of 1 ton? Ans. \$125.

8. If $\frac{5}{8}$ of a barrel of flour costs \$7.50, how much is that a barrel?

9. If 6.5 barrels of potatoes cost \$54.16$\frac{2}{3}$, what is the price per barrel? Ans. 8.33$\frac{1}{3}$.

10. If 17$\frac{1}{2}$ yards of lace cost \$10$\frac{15}{16}$, what is the price per yard?

CASE II.

Art. 265. To find the number of articles, or collections of articles, from their cost and price.

Ex. 1. At $0.125 a pound, how many pounds of sugar can be bought for $25? Ans. 200.

WRITTEN PROCESS.

$0.125)$25.000(200
250
———
00

ANALYSIS.—At $12\frac{1}{2}$ cents a pound, as many pounds can be bought for $25, or 25000 mills, as 125 mills is contained times in 25000 mills, that is, 200 pounds.

Another analysis, easily used in this example, finds 8 lbs. for $1, and 25 × 8 lbs. for $25.

2. At $17.50 per thousand feet of lumber, how many thousand feet can be bought for $350? Ans. 20.

Rule.—*Divide the cost by the price.*

EXAMPLES FOR PRACTICE.

3. At $.20$\frac{1}{2}$ a pound, how many pounds of butter can be bought for $38.54? Ans. 188.

4. At 1\frac{3}{4}$ a bushel, how many bushels of wheat can be bought for 48.12\frac{1}{2}$? Ans. 27$\frac{1}{2}$.

5. At $.16$\frac{2}{3}$ a dozen, how many dozen eggs can you buy for $8?

6. At $.42 a pound, how many pounds of wool can be bought for $161.49?

7. At $\frac{3}{5}$ of a cent apiece, how many apples can you buy for $$\frac{3}{5}$? Ans. 100.

COUNTING ROOM EXERCISES.

Art. 266. In addding long columns, such as often occur in business, the learner should seek to acquire the habit of adding *rapidly*, as well as accurately.

(1.)	(2.)	(3.)	(4.)
$ 43.25	$402.43	$2312.76	$23168.47
37.46	263.48	4128.71	34762.95
12.52	73.19	1024.52	8754.63
20.83	84.45	1645.37	6200.80
19.47	92.17	802.28	7008.97
143.62	17.46	144.44	82756.48
287.70	25.25	5162.78	54275.16
60.00	36.36	1728.31	21076.10
400.60	162.40	14.60	32189.60
78.56	208.28	678.92	6175.38
65.41	605.65	519.79	806.54
300.06	82.00	2178.57	5200.06
60.12	500.00	806.80	97642.87
110.10	77.16	50.00	21435.38
205.46	96.44	7654.75	34276.41
180.34	166.74	8421.77	5026.50
46.35	43.56	749.23	600.89
79.97	384.85	971.18	7684.17
8.40	65.28	9876.54	56827.68

Synopsis of Operations in U. S. Money.

OPERATIONS.	PURPOSES.	APPLICATIONS.
Notation.		
Numeration.		
Reduction.	To lower denominations. To higher denominations.	
Addition.		
Subtraction.		
Multiplication.	Common. To find cost fr. no. and price. To find cost fr. no. and pr. of sets.	Accounts. Bills.
Division.	To find pr. fr. cost and no. *To* find no. fr. cost and pr.	

CHAPTER XIII.

REDUCTION OF COMPOUND NUMBERS.

Art. 267. Operations in denominate numbers are reduction, addition, subtraction, multiplication, or division.

REDUCTION OF DENOMINATE INTEGERS.

Art. 268. **Reduction of denominate numbers** is the process of changing them from one denomination to another. This does not alter their value. It is of two kinds, namely, *descending*, and *ascending*.

Descending Reduction is changing a number to a lower denomination. It is performed by multiplication.

Ascending Reduction is changing a number to a higher denomination. It is performed by division.

CASE I.

Art. 269. To reduce an integer to a lower denomination.

Ex. 1. How many rods are 25 miles? Ans. 8000.

WRITTEN PROCESS.

```
  25  miles.
   8  fur. in 1 mi.
 ----
 200  fur. in 25 mi.
  40  rd. in 1 fur.
 ----
8000  rd. in 25 mi.
```

EXPLANATION.

Since 1 mile is 8 fur., 25 mi. are 25 times 8 furlongs, or 200 fur. Since 1 fur is 40 rd., 200 fur. are 200 times 40 rd., or 8000 rd.

NOTE.—The following simple process obtains the result:—Since 1 mile is 320 rods, 25 miles are 25 times 320 rods, or 8000 rods.

Ex. 2. Reduce 17 mi. 5 fur. 19 rd. 8 ft. 4 in. to inches.
Ans. 1120582 inches.

First Method.

17 mi. 5 fur. 19 rd. 8 ft. 4 in.
8 fur. in 1 mi.
141 fur.
40 rd. in 1 fur.
5659 rd.
$16\frac{1}{2}$ ft. in 1 rd.
$2829\frac{1}{2} = \frac{1}{2}$ of 5659.
$90544 = 16 \times 5659$.
8 = ft. given.
$93381\frac{1}{2}$ ft.
12 inches in 1 ft.
$6 = 12 \times \frac{1}{2}$.
$1120572 = 12 \times 93381$.
4 = inches given.
1120582 inches.

Explanation.

Since 1 mi. is 8 fur., 17 mi. are 17 times 8 fur., or 136 fur., which, with its 5 fur. given, makes 141 fur. Since 1 fur. is 40 rd., 141 fur. are 141 times 40 rd., or 5640 rd., which, with the 19 rd. given, makes 5659 rd. Since 1 rd. is $16\frac{1}{2}$ ft., 5659 rd. are 5659 times $16\frac{1}{2}$ ft., or $93373\frac{1}{2}$ ft., which, with the 8 ft. given, makes $93381\frac{1}{2}$ feet. Since 1 foot is 12 in., $93381\frac{1}{2}$ ft. are $93381\frac{1}{2}$ times 12 in., or 1120578 in., which, with its 4 in. given, makes 1120582 inches.

Rule.—*Multiply the number in the highest denomination by that number of the next lower denomination which makes one of the denomination multiplied. If the number to be reduced is compound, add to this product that part of the number which is of the same denomination as the product.*

Change this result in like manner to the next lower denomination, and so on till the required denomination is reached.

Examples for Practice.

3. Reduce 1 mi. 2 fur. 4 yd. 2 ft. to inches.
Ans. 79368 in.
4. In 1 lea. how many ft.?
5. What will it cost to inclose a farm $1\frac{1}{4}$ mi. square at \$1.25 per rd.? Ans. \$2000.
6. Reduce 15 French ells 2 qr. to yd. Ans. 23.
7. Change 10 ch. 25 l. to inches.
8. Reduce 4 mi. 18 ch. 3 p. to links.
9. How many sq. ft. in 1 A.?
10. Reduce 3 A. 2 R. 17 P. to P.

11. After selling 45 A. from a farm containing 63 A., how many square rods remain? Ans. 2880.

12. How many yards of carpet 1 yd. wide will be required to cover the floor of a room 11 yd. long and 8 yd. wide?

13. In 13 A. 6 sq. ch. 10 P. 105 sq. l. how many square links?

14. In 43 cu. ft. how many cu. in.? Ans. 74304.

15. How many cubic feet in a cellar 15 ft. long, 12 ft. wide, and 5.5 feet deep? Ans. 990.

16. How many cubic yards in a room 12 yd. long, 9.5 yd. wide, and $4\frac{2}{3}$ yd. high? Ans. 532.

17. How many cu. ft. of wood in a pile 96 ft. long, 6 ft. wide, and 5 ft. high? Ans. 2880.

18. How many cubic feet in a wall 123.75 ft. long, $1\frac{1}{2}$ ft. thick, and 4 ft. high? Ans. 742.5.

19. How many feet in a 1 inch board 18 ft. long, and averaging 18 inches wide? Ans. 27.

20. In a plank 15 feet long, 18 inches wide, and 2 inches thick, how many board feet? Ans. 45.

21. How many board feet in a log 16 feet long, and averaging 15 inches square? Ans. 300.

22. Reduce 5 gal. 3 qt. 1 pt. to gills. Ans. 188 gi.

23. In 2 bu. 2 pk. 2 qt. 1 pt. how many pints? Ans. 165 pt.

24. Reduce 3 lb. 2 oz. 5 pwt. 7 gr. to grains. Ans. 18367 gr.

25. Reduce 2 lb. 2 ℥ 1 ʒ 2 ℈ to grains. Ans. 12580 gr.

26. Reduce 3 cwt. 4 lb. 10 oz. to drams.

27. Reduce 5° 25′ 18″ to seconds.

28. Reduce 2 da. 7 hr. 20 min. to seconds.

29. Reduce £5 16 s. 6 d. to farthings. Ans. 5592 far.

30. How many steps of 2 ft. 8 in. each, will a man take, in walking from Richmond to Petersburg, the distance being 18 miles?

31. B. bought 12 bu. of chestnuts at $4.25 a bushel, and retailed them at $.08$\frac{1}{3}$ a pint; how much did he gain on the whole?

32. How many feet high is a horse that measures 17 hands?

33. Reduce 4 O. 5 f℥. 4 fʒ. 24 ♏ to minims.

34. Bought 2 T. 45 lb. of cheese for $400, and retailed it at 16 cts per lb.; what did I gain? Ans. $247.20.

35. What cost 3 cwt. 24 lb. of beef at $.06¾ per pound?

36. Reduce 2 lb. 5 ℥. 4 ʒ. to scruples.

37. Reduce 8 M.M. to M. Ans. 8000 M.

38. Reduce 3.625 K.M. to M.

39. Reduce 6.05 M.^{2} to d.m.^{2} Ans. 605 d.m.^{2}

40. Reduce 4.00138 M.^{2} to m.m.^{2}

41. Reduce 4.324 M.^{3} to d.m.^{3} Ans. 4324 d.m.^{3}

42. Reduce 8 M.^{3} to c.m.^{3}

43. Reduce 7.321 H.L. to L. Ans. 732.1 L.

44. Reduce 5.55 d.l. to m.l.

45. Reduce 15 G. to m.g. Ans. 15000 m.g.

46. Reduce 2.165 K.G. to G.

CASE II.

Art. 270. To reduce an integer to a higher denomination.

Ex. 1. Reduce 1120582 inches to miles.

Ans. 17 mi. 5 fur. 19 rd. 8 ft. 4 in.

WRITTEN PROCESS.

```
12)1120582 in.
 16½ | 93381 ft. 10 in.
  2  |     2
 33)186762
     40)5659 rd. 15/2 ft.=7½ ft.
          8)141 fur. 19 rd.
             17 mi. 5 fur.
```

17 mi. fur. 19 rd. 7½ ft. 10 in.
= 17 mi. 5 fur. 19 rd. 8 ft. 4 in.

EXPLANATION.

In 1120582 in. there are as many ft. as 12 in. are contained times in 1120582 in., namely, 93381 ft. and 10 in. remain. In 93381 ft. there are as many rd. as 16½ ft. are contained times in 93381 ft.; or as many rd. as 33 half ft. are contained times in 186762 half ft., namely, 5659 rd., and 15 half ft. remain. In 5659 rd. there are as many fur. as 40 rd. are contained times in 5659, namely, 141 fur., and 19 rd. remain. In 141 fur. there are as many miles as 8 fur. are contained times in 141 fur., namely, 17 mi., and 5 fur. remain. Hence 1120582 in. equal 17 mi. 5 fur. 19 rd. 7½ ft. 10 in. But ½ ft. is 6 in., which, added to 10 in., makes 16 in., or 1 ft. 4 in. The 1 ft. added to 7 ft. makes 8 ft.

Rule.—*Divide the given number by that number of its denomination which makes one of the next higher. Change this quotient in like manner to the next higher denomination, and so continue till the required denomination is reached. The answer is composed of the last quotient and all the remainders.*

NOTE 1.—Each remainder should be marked as being of the same denomination as that dividend from which it came.

NOTE 2.—Descending and ascending reduction may each be used as a proof for the correctness of the other.

EXAMPLES FOR PRACTICE.

2. Reduce 151338 in. to miles.
Ans. 2 mi. 3 fur. 4 rd. 5 ft. 6 in.

3. In 63360 ft. how many lea.? Ans. 3.

4. Reduce 4321 ft. to fur.
Ans. 6 fur. 21 rd. 4 yd. 2 ft. 6 in.

5. Reduce 8118 in. to ch. Ans. 41 ch.

6. Reduce 13 Ell E. 3 qr. to yd. Ans. 10 yd. 2 qr.

7. How many acres in a field 51 rd. long and 42 rd. wide?

8. How many acres in a field containing 304920 sq. ft.?
Ans. 7.

9. Reduce 457 P. to A. Ans. 2 A. 3 R. 17 P.

10. How many yards of carpet 1 yd. wide will be required to cover the floor of a room 33 ft. long and 24 feet wide?
Ans. 88.

11. How many cubic yards in 124560 cu. in.?

12. What will it cost to dig a cellar 15 ft. long, 12 ft. wide, and 6 ft. deep, at $4.50 per cu. yd.? Ans. $180.

13. What will be the cost of flooring a hall 84 ft. long and 36 feet wide, at $12 per square? Ans. $362.88.

14. What is the cost of a pile of bark 32 ft. long, 12 ft. wide, and 8 ft. high, at $10.24 per cord? Ans. $245.76.

15. What will it cost to build a wall 74.25 ft. long, 1.75 thick, and 6.4 ft. high, at $5.60 a perch? Ans. $188.16.

16. A field is 60 rd. long; how wide must it be to contain 18 acres? Ans. 48 rd.

17. A room is 25 ft. long, 15 ft. wide, and contains 4875 cu. ft.; how high is it? Ans. 13 ft.

18. In 190 gi. how many gallons?

Ans. 5 gal. 3 qt. 1 pt. 2 gi.

19. Reduce 319 pt. to bushels.

Ans. 4 bu. 3 pk. 7 qt. 1 pt.

20. Reduce 18751 gr. Troy to pounds.

21. Reduce 13527 gr. to ℥.

22. Reduce 75436 lb. to tons.

23. Reduce $14770''$ to degrees. Ans. $4° 6' 10''$.

24. In 243725 sec. how many days?

Ans. 2 da. 19 hr. 4 min. 5 sec.

25. At $\$.01\frac{1}{4}$ per pound, how much bran can be bought for $84.60? Ans. 3 T. 7 cwt. 68 lb.

26. At $\$.01\frac{1}{3}$ per grain, how much quinine can be bought for $15.60? Ans. 2 ℥. 3 ʒ. 1 ℈. 1 gr.

27. How much time will a person gain in 30 years by rising 25 min. earlier each day, reckoning 7 leap-years?

Ans. 190 da. 5 hr. 25 min.

28. What will be the cost of 157 yds. of muslin, at 8d. per yard? Ans. £5. 4s. 8d.

29. What will be the cost of paving a sidewalk 60 ft. long, and 12 ft. wide, at $1.25 per square yard? Ans. $100.

30. Reduce 3423 A. to H.A. Ans. 34.23 H.A.

31. Reduce 4127 d.m.^{3} to M.^{3} Ans. 4.127 M.^{3}

32. Reduce 350 c.l. to L. Ans. 3.5 L.

33. Reduce 74 G. to D.G.

34. Reduce 17428 m.g. to G.

35. How many reams of paper in 1575 sheets?

36. Change 79 gi. to gallons.

37. How many bushels in 255 pt. of beans?

38. A miner sold gold-dust at 3 cents a grain; how many pounds had he, if it amounted to $910.80?

Ans. 5 lb. 3 oz. 5 pwt.

REDUCTION OF DENOMINATE FRACTIONS.

Art. 271. **A denominate fraction** is a fraction which expresses one or more of the equal parts of a denominate

unit. Thus, $\frac{1}{2}$ *of a foot,* $\frac{3}{4}$ *of an ounce,* $\frac{2}{3}$ *of a minute,* and $\frac{4}{5}$ *of a degree,* are denominate fractions.

Reduction of denominate fractions is changing them to other denominations. It is finding the value of denominate fractions, expressed either as fractions or integers of other denominations. It is of two kinds, namely, *descending* and *ascending.*

Case I.

Art. 272. To reduce a fraction to a lower denomination.

Ex. 1. What part of a pint is $\frac{1}{80}$ of a bu.? Ans. $\frac{4}{5}$ pt.

First Process Indicated.

$$\frac{1}{\underset{5}{\cancel{80}}}\times\frac{4}{1}\times\frac{\cancel{8}}{1}\times\frac{\cancel{2}}{1}=\frac{4}{5}$$

Second Process Indicated.

$$\frac{1}{80}\times\frac{64}{1}=\frac{64}{80}=\frac{4}{5}.$$

Explanation.

One-eightieth of 1 bu. is $\frac{1}{80}$ of 4 pk. = $\frac{4}{80}$ of 1 pk., = $\frac{4}{80}$ of 8 qt., = $\frac{32}{80}$ of 1 qt., = $\frac{32}{80}$ of 2 pt., = $\frac{64}{80}$ of 1 pt., = $\frac{4}{5}$ of 1 pt.

Note. — The above separate steps of analysis may be represented in one continued product, and the work shortened by cancellation, as shown in the written process.

In the Second Process, $\frac{1}{80}$ bu. = $\frac{1}{80}$ of 64 pt. = $\frac{64}{80}$ pt., = $\frac{4}{5}$ pt.

Rule.—*Multiply the fraction by the numbers that reduce its denomination to the denomination required.*

Examples for Practice.

2. What part of a yard is .04 rd.? Ans. .22.

Process Indicated.—.04 rd. = .04 × 5.5 yd. = .22 yd.

What part

3. Of 1 rd. is $\frac{1}{480}$ mi.? Ans. $\frac{2}{3}$.
4. Of 1 ft. is $\frac{1}{264}$ rd.? Ans. $\frac{1}{16}$.
5. Of 1 in. is $\frac{1}{45}$ yd.?
6. Of 1 l. is $\frac{1}{120}$ ch.? Ans. $\frac{5}{6}$.
7. Of 1 cu. ft. is $\frac{7}{243}$ cu. yd.?
8. Of 1 pt. is $\frac{5}{72}$ gal.? Ans. $\frac{5}{9}$.

What part

9. Of 1 pt. is $\frac{7}{704}$ bu.? Ans. $\frac{7}{11}$.
10. Of 1 pwt. is $\frac{11}{2880}$ lb.?
11. Of 1 ʒ. is $\frac{1}{240}$ lb.? Ans. $\frac{2}{5}$.
12. Of 1 cwt. is $\frac{3}{160}$ T.? Ans. $\frac{3}{8}$.
13. Of 1 min. is $\frac{1}{7200}$ da.?
14. Of 1 qt. is .09 gal.? Ans. .36

15. Reduce .0137 bu. to the decimal of a pint.
Ans. .8768.

16. Reduce .0017 rd. to the decimal of an inch.

17. Reduce $\frac{13}{2560}$ of an acre to the fraction of a perch.
Ans. $\frac{13}{16}$.

18. Reduce .00041 mi. to the decimal of a rod.
Ans. .1312.

19. Reduce $\frac{1}{45}$ of a rod to the fraction of a link.

20. Reduce $\frac{1}{4224}$ of a mile to the fraction of a yard.
Ans. $\frac{5}{12}$.

21. Reduce .007 of a degree to the decimal of a minute.
Ans. .42.

22. Reduce .043 of a scruple to the decimal of a grain.

CASE II.

Art. 273. To reduce a fraction to a higher denomination.

Ex. 1. What part of a day, is $\frac{4}{5}$ of 1 min.?
Ans. $\frac{1}{1800}$ da.

FIRST PROCESS INDICATED.

$$\frac{\cancel{4}}{5}\times\frac{1}{60}\times\frac{1}{\underset{6}{\cancel{24}}}=\frac{1}{1800}$$

SECOND PROCESS INDICATED.

$$\frac{\cancel{4}}{5}\times\frac{1}{\underset{360}{\cancel{144}.\cancel{0}}}=\frac{1}{5\times360}=\frac{1}{1800}$$

$\frac{4}{5}$ min. $=\frac{4}{5}$ of $\frac{1}{1440}$ da., $=\frac{1}{1800}$ da.

EXPLANATION.

Four-fifths of 1 min. $=\frac{4}{5}$ of $\frac{1}{60}$ hr., $=\frac{4}{300}$ hr., $=\frac{4}{300}$ of $\frac{1}{24}$ da., $=\frac{4}{7200}$ da., $=\frac{1}{1800}$ da.

NOTE.—The above separate steps of analysis may be represented by one continued product, and the work shortened by cancellation, as in the written process.

In the Second Process,

Rule.—*Divide the fraction by the numbers that reduce its denomination to the denomination required.*

EXAMPLES FOR PRACTICE.

2. What part of a rod is .22 yd.? Ans. .04 rd.

PROCESS INDICATED.—.22 yd. = .22 of ($\frac{1}{5.5}$ rd. =) $\frac{2}{11}$ rd. = $\frac{.44}{11}$ rd. = .04 rd.

What part

3. Of 1 mi. is $\frac{2}{3}$ rd.? Ans. $\frac{1}{480}$.
4. Of 1 rd. is $\frac{3}{4}$ ft.? Ans. $\frac{1}{22}$.
5. Of 1 yd. is $\frac{4}{5}$ in.? Ans. $\frac{1}{45}$.
6. Of 1 ch. is $\frac{5}{8}$ l.?
7. Of 1 cu. yd. is $\frac{7}{9}$ cu. ft.?
8. Of 1 gal. is $\frac{5}{9}$ pt.?
9. Of 1 bu. is $\frac{7}{11}$ pt.? Ans. $\frac{7}{704}$.
10. Of 1 A. is $\frac{4}{5}$ P.? Ans. $\frac{1}{200}$.
11. Of 1 ℔. is $\frac{9}{10}$ ℈?
12. Of 1 da. is $\frac{1}{2400}$ min.?
13. Of 1 lb. is $\frac{16}{25}$ pwt.?
14. Of 1 gal. is .36 qt.?

15. Reduce .8768 of a pint to the decimal of a bushel. Ans. .0137.

16. Reduce .3366 in. to the decimal of a rod. Ans. .0017.

17. Reduce $\frac{13}{16}$ P. to the fraction of an acre. Ans. $\frac{13}{2560}$.

18. Reduce .1312 rd. to the decimal of a mile.

19. Reduce $\frac{5}{8}$ of a link to the fraction of a rod.

20. Reduce $\frac{5}{12}$ of a yard to the fraction of a mile. Ans. $\frac{1}{4224}$.

21. Reduce .048 gr. to the decimal of a dram. Ans. .0008.

CASE III.

Art. 274. To reduce a denominate fraction to integers of a lower denomination, or, to find its value.

Ex. 1. What is the value of $\frac{2}{3}$ of a bushel? Ans. 2 pk. 5 qt. $0\frac{2}{3}$ pt.

PROCESS INDICATED.

$\frac{2}{3}$ bu. = $\frac{2}{3}$ of 4 pk., = $\frac{8}{3}$ pk., = $2\frac{2}{3}$ pk.
$\frac{2}{3}$ pk. = $\frac{2}{3}$ of 8 qt., = $\frac{16}{3}$ qt., = $5\frac{1}{3}$ pt.
$\frac{1}{3}$ qt. = $\frac{1}{3}$ of 2 pt., = $\frac{2}{3}$ pt., = $0\frac{2}{3}$ pt.

Ex. 2. What is the value of £.580375? Ans. 11s. 7d. 1.16 far.

Written Process.

$$
\begin{array}{l}
£.5\,8\,0\,3\,7\,5 \\
\underline{\qquad\qquad 2\,0} \\
1\,1.6\,0\,7\,5\,0\,0 \text{ shillings.} \\
\underline{\qquad\quad 1\,2} \\
7.2\,9\,0\,0 \text{ pence.} \\
\underline{\qquad 4} \\
1.1\,6 \text{ farthings.}
\end{array}
$$

Analysis.

Five hundred eighty thousand three hundred seventy-five millionths of a pound sterling = .580375 of 20 shillings, = 11.6075 shillings: .6075 of a shilling = .6075 of 12 pence, = 7.29 pence: .29 of a penny = .29 of 4 farthings, = 1.16 farthings. Therefore £.580375 = 11s. 7d. 1.16 far.

Rule.—*Reduce the fraction to the first denomination in which its value will be a whole or mixed number. If its value is a mixed number, reduce its fractional part in the same manner, and so on. The answer consists of the products thus obtained.*

Examples for Practice.

What is the value of

3. $\frac{7}{11}$ of a mile? Ans. 5 fur. 3 rd. 3 yd. 1 ft. 6 in.
4. $\frac{13}{16}$ of an acre? Ans. 3 R. 10 P.
5. .125 of a link? Ans. 9.9 in.
6. $\frac{7}{12}$ of a square rod?
   Ans. 17 sq. yd. 5 sq. ft. 117 sq. in.
7. .75 of a cord? Ans. 96 cu. ft.
8. $\frac{4}{5}$ of a gallon? Ans. 3 qt. $1\frac{3}{5}$ gi.
9. .984375 of a bushel? Ans. 3 pk. 7 qt. 1 pt.
10. Reduce $\frac{5}{7}$ of a pound Troy to integers of lower denominations.
11. What is the value of .2628359375 of a ton?
12. What is the value of $\frac{371}{576}$ of a pound Apothecaries' Weight?
13. Reduce $\frac{5}{9}$ of a week to integers of lower denominations? Ans. 3 da. 21 hr. 20 min.
14. Reduce .475 of a day to integers of lower denominations.
15. What is the value of $\frac{4}{7}$ of a great gross?
16. What is the value of $\frac{3}{7}$ of a circle?
    Ans. $154° 17' 8\frac{4}{7}''$.

17. What is the value of .4375 of a ream?

Ans. 8 quires 18 sheets.

18. What is the value of $\frac{3}{7}$ of a £? Ans. 8s. 6d. $3\frac{3}{7}$ far.

CASE IV.

Art. 275. To find the part that one denominate number is of another.

Ex. 1. What part of a week are 3 hr. 20 min. 40 sec.?

Ans. $\frac{43}{2160}$, $= .0199\dot{0}7\dot{4}$.

PROCESS INDICATED. EXPLANATION.

$$\frac{3 \text{ hr. } 20 \text{ min. } 40 \text{ sec.}}{7 \text{ days.}} = \frac{12040 \text{ sec.}}{604800 \text{ sec.}} = \frac{43}{2160}.$$

Since things compared must be of the same kind, 3 hr. 20 min. 40 sec. must be that part of a week which 12040, the number of seconds in 3 hr. 20 min. 40 sec. is of 604800, the number of seconds in one week, or 7 days, that is, $\frac{43}{2160}$ of a week.

Ex. 2. What decimal of a week are 3 hr. 20 min. 40 sec.?

Ans. $.0199\dot{0}7\dot{4}$.

ANOTHER PROCESS.

60	40.0000000
60	20.6666666+
24	3.3444444+
7	$0.139\dot{3}51\dot{8}$
	$0.0199\dot{0}7\dot{4}$

EXPLANATION.

Since 60 sec. make 1 min., 40 sec. are $\frac{40}{60}$ of a min., or .6666 + of a min. Since 60 min. make 1 hr., 20.66 + min. are $\frac{1}{60}$ as many hours as minutes, namely, 3.3444 + hr. Since 24 hr. make 1 day, 3.3444 + hr. are $\frac{1}{24}$ as many days as hours, namely, $0.139\dot{3}51\dot{8}$ days. Since 7 days make 1 week, $0.139\dot{3}51\dot{8}$ days are $\frac{1}{7}$ as many weeks as days, namely, $0.0199\dot{0}7\dot{4}$ wk.

Ex. 3. What part of $\frac{1}{2}$ day is $\frac{3}{4}$ hr.? Ans. $\frac{1}{16}$, or .0625.

FIRST METHOD.

$\frac{1}{2}$ day = 12 hr.: $\frac{\frac{3}{4}\text{ hr.}}{12 \text{ hr.}} = \frac{3}{48} = \frac{1}{16} = .0625.$

Second Method.

$$\tfrac{3}{4}\text{ hr.} = \tfrac{3}{4}\text{ of }\tfrac{1}{24}\text{ da.} = \tfrac{3}{96}\text{ da.} = \tfrac{1}{32}\text{ da.} : \frac{\tfrac{1}{32}\text{ da.}}{\tfrac{1}{2}\text{ da.}} = \tfrac{2}{32} = \tfrac{1}{16} = .0625.$$

Third Method.

$$\tfrac{1}{2}\text{ da.} = 720\text{ min.} : \tfrac{3}{4}\text{ hr.} = 45\text{ min.} : \tfrac{45}{720} = \tfrac{1}{16} = .0625.$$

Rule.—*Reduce both numbers to the same denomination; then write that which is to be the part for a numerator, and the other for a denominator.*

Note 1.—When the result is to be a common fraction, it is generally best to reduce it to its lowest terms.

Note 2.—When the result is to be a decimal fraction, reduce the common fraction, formed by the rule, to a decimal.

Examples for Practice.

4. What part of a rod is 2 yd. 1 ft. 10 in.? Ans. $\frac{47}{99}$.
5. What part of a mile is 3 fur. 25 rd.?
6. Reduce 3 R. 28 P. to the decimal of an A. Ans. .925.
7. Reduce 54 ch. 3 p. 15 l. to the decimal of a mile. Ans. .68625.
8. What part of 1 gallon is $\frac{1}{2}$ pint? Ans. $\frac{1}{16}$.
9. What part of 1 cu. yd. is 7 cu. ft. 864 cu. in.? Ans. $\frac{5}{18}$.
10. What part of a day is 2 hr. 15 minutes?
11. 5 bu. 3 pk. 4 qt. 1 pt. = how many bushels? Ans. 5.890625.
12. 6 T. 8 cwt. 15 lb. = how many tons? Ans. 6.4075.
13. £3 15s. 9d. = how many £? Ans. $3\frac{63}{80}$, or 3.7875.
14. What part of $\frac{1}{2}°$ is $\frac{2}{3}'$?
15. What part of \$1 is $7\frac{1}{2}$ cents? Ans. $\frac{3}{40}$.
16. What part of 1° is 45′ 45″? Ans. $\frac{61}{80}$.
17. What decimal of a ream is 10 quires 18 sheets?

ADDITION OF COMPOUND NUMBERS.

Art. 276. To find the sum of two or more compound numbers.

Ex. 1. What is the sum of 5 A 2 R. 34 P. 28 yd. 8 ft. 95 in. + 9 A. 3 R. 21 P. 19 yd. 7 ft. 116 in. + 1 R. 38 P. 29 yd. 8 ft. + 7 P. 16 yd. 6 ft. 53 in.?

Ans. 16 A. 0 R. 23 P. 4 yd. 6 ft. 12 in.

WRITTEN PROCESS.

A.	R.	P.	yd.	ft.	in.
5	2	34	28	8	95
9	3	21	19	7	116
	1	38	29	8	
		7	16	6	53
16	0	23	$4\frac{1}{4}$	3	120
			$\frac{1}{4} =$	$2\frac{1}{4}$	
				$\frac{1}{4} =$	36
16	0	23	4	6	12

EXPLANATION.

The sum of the inches is 264, which equals 1 square foot, and 120 square inches remain. Write 120, and carry 1 to the column of feet, making its sum 30, which equals 3 square yards, and 3 square feet remain. Write 3 ft., and carry 3 yd. to the column of yards, making its sum 95, which, divided by $30\frac{1}{4}$, gives 3 poles, and $4\frac{1}{4}$ yd. remain. Write $4\frac{1}{4}$, and carry 3 to the column of poles, making its sum 103, which, divided by 40, gives 2 roods, and 23 poles remain. Write 23, and carry 2 to the column of roods, making its sum 8, which, divided by 4, gives 2 acres, and no remainder. Carry 2 to the column of acres, making its sum 16.

Now, to express all the denominations in whole numbers, we find the value of $\frac{1}{4}$ yd. $= \frac{1}{4}$ of 9 ft., $= 2\frac{1}{4}$ ft., and the value of $\frac{1}{4}$ ft. $= \frac{1}{4}$ of 144 in., $=$ 36 in. The sum of the inches is now 156, $=$ 1 ft. 12 in. Carry the 1 ft. to the feet, making the sum 6 ft. The other numbers are unchanged.

Rule.—*Write the numbers so that those of the same denomination may be in the same column.*

Add the numbers of the lowest denomination, and divide their sum by that number of their denomination which makes one of the next higher.

Write the remainder under the column added, and add the quotient to the column of the next higher denomination.

Do thus with all the denominations to the highest, of which write the whole sum.

EXAMPLES FOR PRACTICE.

(2.)

£.	s.	d.	far.
3	11	7	2
4	17	9	3
	12	10	1

(3.)

mi.	fur.	rd.	yd.	ft.	in.
7	6	24	4	2	8
10	5	33	3	1	10
6	7	18	1	2	6
25	3	36	4	2	6

4. In four days a man husked 23 bu. 3 pk., 27 bu. 1 pk., 29 bu. 2 pk., and 31 bu. 3 pk. of corn. How much corn did he husk?

5. Add 3 mi. 180 rd. 4 yd. 2 ft., 5 mi. 73 rd. 5 yd. 1 ft. 6 in., 7 mi. 2 yd. 2 ft. 10 in., and 6 mi. 96 rd. 1 ft. 5 in.

6. How much land in 4 lots which contain 2 A. 54 P., 3 A. 88 P., 1 A 120 P., and 6 A. 33.25 P.?

Ans. 13 A. 135.25 P.

7. The contents of four barrels measured 41 gal. 3 qt. 1 pt. 2 gi., 38 gal. 2 qt. 3 gi., 40 gal. 1 pt. 3 gi., and 39 gal. 3 qt. 1 pt. 1 gi. Find the whole contents.

Ans. 160 gal. 2 qt. 1 pt. 1 gi.

8. Add 135 cu. yd. 17 cu. ft. 1236 cu. in., 148 cu. yd. 16 cu. ft. 1068 cu. in., and 156 cu. yd. 23 cu. ft. 864 cu. in.?

9. Sold to A. 105 cords 65 cu. ft. 1216 cu. in. of wood, to B. 86 cords 87 cu. ft. 1475 cu. in., to C. 68 cords 93 cu. ft. 936 cu. in. How much wood did I sell?

Ans. 260 cords 119 cu. ft. 171 cu. in.

10. A farmer sold 4 stacks of hay, weighing 5 T. 13 cwt. 43 lb., 4 T. 15 cwt. 65 lb., 6 T. 9 cwt. 80 lb., and 3 T. 8 cwt. 56 lb., respectively. How much hay did he sell?

Ans. 20 T. 7 cwt. 44 lb.

11. What is the sum of 3 ℔. 7 ℥ 5 ʒ 2 ℈ 15 gr., 5 ℔. 4 ℥ 2 ℈ 12 gr., and 2 ℔. 6 ℥ 3 ʒ 1 ℈ 18 gr.?

Ans. 11 ℔. 6 ℥ 2 ʒ 1 ℈ 5 gr.

12. Add 3 da. 5 hr. 36 min. 45 sec., 5 da. 19 hr. 50 min. 55 sec., and 4 da. 10 hr. 40 min. 30 sec.

13. Add 10 lb. 8 oz. 13 pwt. 19 gr., 8 lb. 6 oz. 16 pwt. 15 *gr., and 6 lb.* 9 oz. 14 pwt. 18 gr.

14. A man bought 5 buckets of butter weighing 18 lb. 10 oz., 16 lb. $12\frac{1}{2}$ oz., 17 lb. $13\frac{3}{4}$ oz., 19 lb. $6\frac{1}{2}$ oz., and 16 lb. 8 oz. How much butter in all? Ans. 89 lb. $2\frac{3}{4}$ oz.

15. St. Louis is 90° 15′ 10″ west of Greenwich, and the Cape of Good Hope is 18° 28′ 45″ east of Greenwich; what is the difference of longitude between the two places?

16. A ship started in 17° 15′ 48″ north latitude, and sailed south to a point in 13° 46′ 34″ south latitude; through how many degrees of latitude did she sail? Ans. 31° 2′ 22″.

17. Add $\frac{2}{3}$s. to £$\frac{3}{5}$. (See Case III. of Reduction.)
Ans. 12s. 8d.

18. Add $\frac{3}{4}$ bu. $\frac{3}{8}$ pk. $\frac{1}{2}$ qt. Ans. 3 pk. 3 qt. 1 pt.

19. To $\frac{5}{16}$ cwt. add $\frac{3}{8}$ lb. Ans. 31 lb. 10 oz.

20. To $\frac{3}{5}$ ℔. add $\frac{4}{5}$ ℥. Ans. 8 ℥.

21. What is the sum of $3\frac{3}{5}$ A. + $2\frac{7}{8}$ A.?
Ans. 6 A. 1 R. 36 P.

22. What is the sum of $4\frac{2}{3}$ cords + $3\frac{7}{8}$ cords?

23. Add $\frac{5}{6}$ T. to $\frac{5}{9}$ cwt. Ans. 17 cwt. 22 lb. 3 oz. $8\frac{8}{9}$ dr.

24. Add $\frac{7}{10}$ of a degree, $\frac{7}{12}$ of a minute, and 10° 15′ 25″.

25. To $\frac{4}{7}$ of a pound add 11 oz. 13 pwt. 18 gr.

26. What is the sum of .645 of a day $\frac{7}{18}$ of an hour?
Ans. 15 hr. 52 min. 8 sec.

27. Add 7.88125 acres to $\frac{2}{3}$ of an acre.

28. What is the sum of 15 bu. 1 pk., $5\frac{3}{8}$ bu., 4 bu. $2\frac{1}{2}$ qt., 6 bu. 2.46875 pk., $\frac{2}{3}$ pk.? Ans. 31 bu. 3 pk. 3 qt. $1\frac{1}{6}$ pt.

29. Add £9 5s. $4\frac{3}{5}$d., £6 $15\frac{3}{5}$s., £4 4.465s., 10s. $6\frac{1}{4}$d., and £$\frac{7}{12}$. Ans. £21 7s. 7d. 2.52 far.

30. Add .375 sq. mi., $\frac{17}{32}$ sq. mi., 32.025 A., and 63 A. 2 R. 17 P. 15 sq. ft. 80 sq. in.

31. Bought at one time 5 ℥ 4 ʒ 2 ℈ 13 gr. of quinine, at another time 6 ℥ 5 ʒ 1 ℈ 17 gr. What did it cost at $2.80 per ℥? Ans. $34.475.

32. Add $\frac{3}{4}$ of a week, $\frac{3}{4}$ of a day, $\frac{3}{4}$ of an hour, and $\frac{3}{4}$ of a minute.

33. Add .341 lb., .65 oz., .215 pwt., and 3.12 gr.

34. Add .2875 gal., 2.56 qt., and $1\frac{2}{3}$ gi.

SUBTRACTION OF COMPOUND NUMBERS.

Art. 277. To find the difference of two compound numbers.

Ex. 1. From 8 lb. 9 oz. 17 pwt. 21 gr. take 3 lb. 4 oz. 18 pwt. 15 gr. Ans. 5 lb. 4 oz. 19 pwt. 6 gr.

WRITTEN PROCESS.

lb.	oz.	pwt.	gr.
8	9	17	21
3	4	18	15
5	4	19	6

FIRST EXPLANATION.

The difference of the grains is 6; write 6 under the grains. Now, since 18 pwt. cannot be taken from 17 pwt., we borrow 1 oz. from the 9 oz., and add its value, 20 pwt., to the 17 pwt., making the minuend 8 lb. 8 oz. 37 pwt. Now, 18 pwt. from 37 pwt. leave 19 pwt. Write 19 under the pwt. Then 4 oz. from 8 oz. leave 4 oz. Write 4 under the oz. Lastly, 3 lb. from 8 lb. leave 5 lb., making the whole difference 5 lb. 4 oz. 19 pwt. 6 gr.

SECOND EXPLANATION.

Taking 15 gr. from 21 gr. leaves 6 gr. Write 6 gr. under the grains. Next, 18 pwt. from 17 pwt., impossible. To make subtraction possible, add 20 pwt. to 17 pwt., making 37 pwt.; then, 18 pwt. from 37 pwt. leave 19 pwt. Write 19 pwt. under the pennyweights. Because we increased the minuend by 20 pwt., equal to 1 oz., we, to keep the difference between the numbers unchanged, add 1 oz. to the 4 oz. of the subtrahend, making 5 oz.: then 5 oz. from 9 oz. leave 4 oz. Lastly, 3 lb. from 8 lb. leave 5 lb., making the whole difference 5 lb. 4 oz. 19 pwt. 6 gr.

Ex. 2. From 2 pecks take 5 quarts. Ans. 1 pk. 3 qt.

WRITTEN PROCESS.

pk.	qt.
2	0
0	5
1	3

EXPLANATION.

Since 5 qt. cannot be taken from 0 qt., we add 8 qt., the value of a peck, to the minuend; then, 5 qt. from 8 qt. leave 3 qt. Write 3 under the qt. Now, either diminish the 2 pk. by 1, and say "0 pk. from 1 pk. leaves 1 pk.," or increase the 0 pk. by 1, and say, "1 pk. from 2 pk. leaves 1 pk.," making the whole difference 1 pk. 3 qt.

Rule.—*Write the parts of the subtrahend under those of the same denomination in the minuend, and draw a line under them.*

Then, beginning with the lowest denomination, subtract in order each lower number from that above it, if possible, and write the difference below.

When any lower number is greater than that above it, add as many units of its kind to the upper number as make one of the next higher denomination; then subtract the lower number, and add 1 to the next lower number before subtracting it.

NOTE.—Instead of adding 1 to the next lower number, some prefer to diminish the next upper number by 1, according to the First Explanation of Ex. 1. This plan is generally thought less convenient in practice, especially where the number to be diminished is 0. In this case we must borrow from the still higher denomination, making the operation inconvenient. To illustrate this, try the plan upon Ex. 3, below.

EXAMPLES FOR PRACTICE.

3. From 4 bu. take 2 pk. 3 qt. 1 pt.

Ans. 3 bu. 1 pk. 4 qt. 1 pt.

4. From 1 lb. 3 ℥ of quinine, a druggist sold 2 ℥ 5 ʒ 2 ℈ 12 gr.; how much remained? Ans. 1 lb. 2 ʒ 8 gr.

5. From a barrel containing 41 gal. 2 qt. of oil, was sold 28 gal. 3 qt. 1 pt.; how much remained?

Ans. 12 gal. 2 qt. 1 pt.

6. A farmer raised 28 bu. 1 pk. 3 qt. 1 pt. of clover-seed. After selling 20 bu. 3 pk. 5 qt. how much had he left?

7. Pittsburgh is 2° 57′ 15″ west from Washington, and Cincinnati is 7° 26′ 45″ west from Washington; what is the difference of longitude between Pittsburgh and Cincinnati?

Ans. 4° 29′ 30″.

8. From 3 common years, 60 days, and 8 hours, take 1 yr. 204 da. 12 hr. 28 min. 36 sec.

Ans. 1 yr. 220 da. 19 hr. 31 min. 24 sec.

9. A stationer bought 15 gross 5 doz. lead pencils, and immediately sold 8 gross 7 doz. 6 of them; how many pencils had he left? Ans. 6 gross 9 doz. 6.

10. From 33° 10′ 25″ take 15° 25′ 33″.

11. A stationer bought 2 reams 8 quires 10 sheets of note-paper, and immediately sold 18 quires 15 sheets; how much remained?

12. After selling 13 lb. 8 oz. of butter from a bucket containing 22 lb. how much remained?

13. From $\frac{3}{5}$ rd. take $\frac{3}{4}$ yd. Ans. 2 yd. 1 ft. $7\frac{4}{5}$ in.

14. Subtract 12 rd. 7 ft. $1\frac{3}{4}$ in. from 25 rd. 1.2 ft.
Ans. 12 rd. 10 ft. 6.65 in.

15. From 5 A. take 1 R. $17\frac{2}{3}$ P. Ans. 4 A. 2 R. $22\frac{1}{3}$ P.

16. From $\frac{5}{9}$ wk. take $\frac{3}{7}$ da.

17. From .75 gross take $\frac{2}{3}$ doz. Ans. 100.

18. From 8.75 bu. take $3\frac{7}{8}$ bu. Ans. 4 bu. 3 pk. 4 qt.

19. From 3 sq. ft. 81.135 sq. in. take $.28\frac{1}{3}$ sq. yd.
Ans. 1 sq. ft. 1.935 sq. in.

20. From $\frac{13}{54}$ cu. yd. take $\frac{143}{192}$ cu. ft.
Ans. 5 cu. ft. 1305 cu. in.

21. From $\frac{7}{13}$ ℥ take $\frac{11}{14}$ ʒ. Ans. 3 ʒ 1 ℈ $11\frac{29}{91}$ gr.

22. A boy having 3 pk. 1 qt. 1 pt. of berries, sold .0625 bu.; how many berries had he left? Ans. 2 pk. 7 qt. 1 pt.

23. From 5 lb. 15 gr. take 5 oz. 15 pwt.

24. Subtract $\frac{3}{7}$s. from £$\frac{7}{45}$.

25. A street is $\frac{5}{9}$ of a mile long; after paving $\frac{5}{6}$ of a furlong, how much remains to be paved?
Ans. 3 fur. 24 rd. 7 ft. 4 in.

26. From $\frac{1}{5}$ of a week subtract .78 of a day.

Art. 278. To find the difference between dates.

Rule.—*Subtract the less number from the greater.*

NOTE.—In doing this both numbers may be expressed cardinally, or both ordinally.

EXAMPLES FOR PRACTICE.

Ex. 1. Milton was born A. D. Dec. 9, 1608, and died Nov. 8, 1675. How old was he at death?
Ans. 66 yr. 10 mo. 29 da.

FIRST METHOD.				SECOND METHOD.		
yr.	mo.	da.		yr.	mo.	da.
1675	11	8		1674	10	7
1608	12	9		1607	11	8
66	10	29		66	10	29

EXPLANATION.

His age was the difference in time between the date of his birth and that of his death. Since the date of his death expresses the greater quantity of time, it must be the minuend, and the date of his birth must be the subtrahend. In the FIRST METHOD we reckon the date *ordinally*, thus:—He was born on the 9th day of the 12th month of the 1608th year of the Christian Era, and he died on the 8th day of the 11th month of the 1675th year of the same era.

In the SECOND METHOD we reckon *cardinally*, that is, the actual time that has elapsed at each date, thus:—At his birth there had elapsed of the era 1607 years, 11 months, and 8 days; and at his death 1674 years 10 months, and 7 days. In both methods we, in subtracting, consider 30 days a month, and 12 months a year.

NOTE.—This method of operating with dates does not obtain the exact difference, because it reckons all years as equal, and all months as equal, whereas they vary.

Art. 279. To find the exact difference of time between two dates, the units must be unvarying. Years and months are of variable length, but *days, hours, minutes,* and *seconds* are uniform.

TABLE,

SHOWING THE NUMBER OF DAYS FROM ANY DAY OF ONE MONTH TO THE SAME DAY OF ANY OTHER MONTH WITHIN A YEAR OF THE FORMER.

FROM ANY DAY OF.	TO THE SAME DAY OF THE NEXT											
	Jan.	Feb.	Mar.	Apr.	May.	June	July.	Aug.	Sept.	Oct.	Nov.	Dec.
January,	365	31	59	90	120	151	181	212	243	273	304	334
February,	334	365	28	59	89	120	150	181	212	242	273	308
March,	306	337	365	31	61	92	122	153	184	214	245	275
April,	275	306	334	365	30	61	91	122	153	183	214	244
May,	245	267	304	335	365	31	61	92	123	153	184	214
June,	214	245	273	304	334	365	30	61	92	122	153	183
July,	184	215	243	274	304	335	365	31	62	92	123	153
August,	153	184	212	243	273	304	334	365	31	61	92	122
September,	122	153	181	212	242	273	303	334	365	30	61	91
October,	92	123	151	182	212	243	273	304	335	365	31	61
November,	61	92	120	151	181	212	242	273	304	334	365	30
December,	31	62	90	121	151	182	212	243	274	304	335	365

Ex. 1. How many days from Aug. 9 to July 4 of the next year?

SOLUTION.—We find August on the left, and trace its line into the column headed July, where we find 334. This means that there are 334 days from Aug. 9 to the next July 9: therefore July 4 must be 5 days less: 334 − 5 = 329, Ans. If an example includes the end of February in leap-year, its answer must have one more day than the number given by the table.

NOTE.—This table is much used in calculating interest, and in other business.

EXAMPLES FOR PRACTICE.

Ex. 2. What was the exact amount of time from A. D. 1867, May 3, at 9 o'c. 15 min. 22 sec. P. M. to April 2, 1868, at 6 o'c. 5 min. 45 sec. A. M.?

Ans. 334 da. 8 hr. 50 min. 23 sec.

SOLUTION.

BY THE TABLE.

Fr. May 3, at 9: 15, 22	is 335 da.
To Apr. 3, at 9: 15, 22	1. yr. 1 da.
	336
Less 1 day to Apr. 2, =	335

9 o'c. 15 m. 22 s. P. M. =	21 hr. 15 min. 22 sec.
6 o'c. 5 m. 45 s. A.M. =	6 hr. 5 min. 45 sec.
Difference	15 hr. 9 min. 37 sec.

335 days — 15 hr. 9 min. 37 sec. = 334 da. 8 hr. 50 min. 23 sec.

WITHOUT THE TABLE.

12 hr.	0 min.	0 sec.	
9	15	22	
2 hr.	44 min.	38 sec.,	rem. in May 3.
6	5	45	in Apr. 2.
8 hr.	50 min.	23 sec.,	+ the days.

The days are, in May 27, June 30, July 31, Aug. 31, Sept. 30, Oct. 31, Nov. 30, Dec. 31, Jan. 31, Feb. 29, March 31, April 2; in all, 334 da. 8 hr. 50 min. 23 sec.

3. Washington was born Feb. 22, 1732, and the United States declared their independence July 4, 1776; how old was Washington at that time? Ans 44 yr. 4 mo. 12 da.

4. The battle of Bunker Hill was fought June 19, 1775, and Fort Sumter was attacked April 12, 1861; what time *elapsed* between those two dates?

5. How long has a note to run, dated Sept. 18, 1867, and made payable July 10, 1870? Ans. 2 yr. 9 mo. 22 da.

6. What time elapsed from 6 o'clock P. M. of Aug. 10, 1864, to 8 o'clock A. M. of May 18, 1871?
Ans. 6 yr. 9 mo. 7 da. 14 hr.

7. How many days from the 4th of March, to the 25th of December following? Ans. 296.

8. How many days from July 4th, to Christmas following?

9. How many days from Sept. 10, 1867 to March 5, 1872?
Ans. 1638.

10. A man was born Feb. 29, 1820, and died July 29, 1868; what was his age, and how many birthdays had he?
Ans. Age 48 yr. 5 mo.; birthdays 13.

11. How many days from Mar. 8th to Dec. 5th?

MULTIPLICATION OF COMPOUND NUMBERS.

Art. 280. To multiply a compound number by a simple number.

Ex. 1. Multiply 12 lb. 7 oz. 15 pwt. 10 gr. by 5.
Ans. 63 lb. 2 oz. 17 pwt. 2 gr.

WRITTEN PROCESS.

lb.	oz.	pwt.	gr.
12	7	15	10
			5
63	2	17	2

EXPLANATION.

Five times 10 gr. are 50 gr., equal to 2 pwt. 2 gr. Write 2 gr. under the grains, and carry the 2 pwt. to the next product. Five times 15 pwt. are 75 pwt., and the 2 carried make 77 pwt., equal to 3 oz. and 17 pwt. Write the 17 pwt., and carry the 3 oz. to the next product. Five times 7 oz. are 35 oz., and the 3 carried make 38 oz., equal to 3 lb. 2 oz. Write the 2 oz., and carry the 3 lb. to the product of pounds. Five times 12 lb. are 60 lb., and the 3 lb. carried make 63 lb., making the whole product 63 lb. 2 oz. 17 pwt. 2 gr.

Rule.—*Multiply first the number in the lowest denomination, and divide this product by that number of this denomination*

which makes one of the next higher. Write the remainder, and add the quotient to the next product. Do this through all the denominations.

NOTE.—To multiply a compound number by a fraction, either proceed by the above rule, or multiply the compound number first by the numerator, and divide the result by the denominator; or, divide the compound number first by the denominator, and multiply the result by the numerator. These last methods imply a knowledge of division of compound numbers.

EXAMPLES FOR PRACTICE.

(2.)

T.	cwt.	lb.	oz.
5	6	27	8
			9
47	16	47	8

(3.)

bu.	pk.	qt.	pt.
9	2	6	1
			11
106	2	7	1

(4.)

lb.	oz.	pwt.	gr.
7	6	9	13
			12

(5.)

lb.	℥.	ʒ.	℈.	gr.
10	8	7	2	17
				8

6. Multiply 13 cd. 5 cd. ft. 12 cu. ft. by 13.

7. Multiply 6 gal. 2 qt. 1 pt. 2.25 gi. by 64.
Ans. 424 gal. 2 qt.

8. Multiply 7 cu. yd. 13 cu. ft. 1485 cu. in. by 36.

9. What is the weight of 7 bales of hay, each weighing 4 cwt. 49 lb. 10 oz.? Ans. 1 T. 11 cwt. 47 lb. 6 oz.

10. Multiply 12 hr. 36 min. 45 sec. by $10\frac{2}{3}$.

11. How much castor oil in 75 bottles, each containing 4 f℥ 4 fʒ 23 ♏? Ans. 2 Cong. 5 O. 5 f℥ 45 ♏.

12. Multiply 8 A. 3 R. 15 P. 137 sq. ft. 100 sq. in. by 40.
Ans. 353 A. 3 R. 20 P. 62 sq. ft. 112 sq. in.

13. Multiply 3 yd. 2 ft. 1.13 in. by 83.
Ans. 306 yd. 2 ft. 9.79 in.

14. Multiply 3 pk. 3 qt. 1 pt. 1.7 gi. by 15.

15. Multiply 2 bu. 3 pk. 7 qt. $1\frac{1}{4}$ pt. by 10.8724.

SUGGESTION.—Reduce to pints, multiply, then reduce the product to the required denomination; or, reduce the lower denominations to the decimal of a bushel, multiply, and reduce the product to the required denomination.

Ans. 32 bu. 1 pk. 7 qt. 1.3465 pt.

16. At the uniform rate of 3 mi. 5 fur. 25 rd. per hour, how far will a man walk in 6.4 hours?

17. Multiply 10° 28′ $42\frac{1}{2}$″ by 2.754.

Ans. 28° 51′ 27.765″.

Art. 281. The **meridian** of any place is a north-and-south line passing through that place. If it is continued around the Earth, it becomes a *great circle* of the Earth, and its plane cuts the Earth into halves from pole to pole. The plane of an observer's meridian divides his visible heavens into eastern and western halves.

Civil time is, usually, solar mean time. (See Notes 2 and 3, Art. 237.) An hour, as marked by a time-piece which keeps mean time, is $\frac{1}{24}$ of the length of an average solar day, that is, $\frac{1}{24}$ of the average time between two consecutive passages of the centre of the Sun over the meridian of the place which keeps the time. Now, as the rotation of the Earth on its axis from west to east causes this phenomenon by bringing the meridian of a place round to the Sun again, and as the rotation is uniform, the Earth must turn, or the Sun seem to move, on the average, $\frac{1}{24}$ of 360°, or 15°, in 1 hour; $\frac{1}{60}$ of 15°, or 15′, in 1 minute of time; and $\frac{1}{60}$ of 15′, or 15″, in 1 second of time.

Apparent noon at any place is the instant when the centre of the Sun is on the meridian of that place. **Mean noon** at any place is the instant when the centre of the Sun would be on the meridian of that place if he actually moved with his average motion. These noons are nearly coincident Apr. 15, June 15, Aug. 31, and Dec. 24. On Feb. 10 apparent noon is about $14\frac{1}{2}$ minutes slower than mean; on

May 15 it is about 3 min. 52 sec. ahead of mean; on July 26 it is about 6 min. 13 sec. slower than mean; and on Nov. 2 it is about 16 min. 19 sec. ahead of mean noon.

Noon in civil time is marked 12 M. for meridian.

Art. 282. To find the difference of longitude between two places, from the difference of their time.

Ex. 1. If a correct chronometer, set to Greenwich time, shows 11 o'c. 5 min. 30 sec. P. M., when a correct clock at Albany shows 6 o'c. 10 min. 31.4 sec. P. M., how many degrees of longitude would that show that Albany is west of Greenwich? Ans. 73° 44′ 39″.

WRITTEN PROCESS.

11 hr.	5 min.	30 sec.	
6	10	31.4	
4 hr.	54 min.	58.6 sec.	= diff. of time.
		15	multiplier.
73°	44′	39″	= diff. of longitude.

EXPLANATION.

When the Sun had passed Albany 6 hr. 10 min. 31.4 sec., it had passed Greenwich 11 hr. 5 min. 30 sec. Therefore Albany must be west of Greenwich, and Albany time is 4 hr. 54 min. 58.6 sec. behind Greenwich time. Since the seconds of arc are 15 times as many as the seconds of time, and the minutes of arc are 15 times as many as the minutes of time, and the degrees 15 times as many as the hours, the difference of longitude is found by multiplying the difference of time by 15, according to the following

Rule.—*Multiply the difference of time by 15, and call the product of seconds of time* seconds of arc, *of minutes of time* minutes of arc, *and of hours* degrees.

EXAMPLES FOR PRACTICE.

2. The difference of time between two places is 3 hr. 24 min. 27 sec.; what is their difference of longitude.

Ans. 51° 6′ 45″.

3. The sun rises at Boston 1 hr. 16 min. 48 sec. sooner than at St. Louis; the longitude of Boston being 71° 3′ W., *what is the* longitude of St. Louis? Ans. 90° 15′ W.

4. When it is 7 o'clock at Philadelphia, it is 6 hr. 27 min. 15 sec. at Columbus; the longitude of Philadelphia being 75° 10′ W., what is the longitude of Columbus?

Ans. 83° 21′ 15″ W.

5. The longitude of Halifax is 63° 36′ 40″ W., and when it is 10 o'clock A. M. at Halifax, it is 8 o'clock 24 min. $24\frac{2}{3}$ sec. at Chicago; what is the longitude of Chicago?

Ans. 87° 30′ 30″ W.

6. The difference of time between New York and Jerusalem is 7 hr. 18 min. $12\frac{2}{5}$ sec.; what is the difference in longitude?

7. When it is 12 o'clock M. at Washington, it is 5 hr. 17 min. $7\frac{7}{15}$ sec. P. M. at Paris; the longitude of Paris being 2° 20′ 22″ E., what is the longitude of Washington?

Ans. 77° 1′ W.

8. A man travelling from Constantinople, which is in 28° 59′ E. longitude, finds, on his arrival at Madras, in India, that his watch is 3 hr. 25 min. $7\frac{4}{5}$ sec. slower than true time at the latter place. If his watch did not vary, what is the longitude of Madras? Ans. 80° 15′ 57″ E.

9. When it is 10 o'clock 15 min. 30 sec. A. M. at Baltimore, it is 5 o'clock 23 min. 17 sec. P. M. at St. Petersburg; the longitude of Baltimore being 76° 37′ W., what is the longitude of St. Petersburg? Ans. 30° 19′ 45″ E.

10. A ship's chronometer, set at Greenwich, points to 7 hr. 13 min. 38 sec. P. M., when the sun is on the meridian; what is the longitude of the ship? Ans. 108° 24′ 30″ W.

11. When it is 9 o'clock 10 min. 20 sec. P. M. at Wheeling, it is 3 o'clock 45 min. 22 sec. A. M. of the following day, at Stockholm; the longitude of Wheeling being 80° 42′ W., what is the longitude of Stockholm? Ans. 18° 3′ 30″ E.

12. I left Cape of Good Hope, longitude 18° 28′ 45″ E., with the true time of that place, and travelled to Canton, where I found my watch to be 6 hr. 19 min. 1 sec. slower than the true time of the place; what is the longitude of Canton? Ans. 113° 14′ E.

DIVISION OF COMPOUND NUMBERS.

Case I.

Art. 283. To divide a compound number by a denominate number of the same kind.

Ex. 1. How many times will a bin, holding 56 pk. 7 qt., contain a measure holding 8 pk. 1 qt.? Ans. 7 times.

By Inspection.

8 pk. 1 qt.) 56 pk. 7 qt. (7

Here we see at once that 1 qt. is contained in 7 qt. the same number of times that 8 pk. is contained in 56 pk., that is, 7 times. Hence the whole divisor is contained in the whole dividend 7 times.

Ex. 2. If the average daily motion of the Moon is 13° 10′ 35″, in how many days would she move 94° 52′ 12″?

Ans. $7\frac{1}{5}$.

First Method.— $$\frac{94°\ 52'\ 12''}{13°\ 10'\ 35''} = \frac{341532''}{47435''} = 7\tfrac{1}{5}.$$

Second Method.— $$\frac{94°\ 52'\ 12''}{13°\ 10'\ 35''} = \frac{94.87°}{13.17638°} = 7\tfrac{1}{5}.$$

Third Method.— $$\begin{array}{r} 13°\ 10'\ 35'')\ 94°\ 52'\ 12''(7\frac{2°\ 38'\ 7''}{13°\ 10'\ 35''} = 7\tfrac{1}{5}. \\ 92\ \ 14\ \ \ 5\qquad\qquad\qquad\qquad \\ \hline 2°\ 38'\ \ 7''\qquad\qquad\qquad\qquad \end{array}$$

Rule.—*When the number of times which the divisor is contained in the dividend cannot be found by inspection,*

I.—*Reduce both to the lowest denomination in either of them; then divide as in simple division; or,*

II.—*Reduce both to the highest denomination in either of them; then divide as in mixed numbers; or,*

III.—*After the highest denomination of the dividend is made the same as the highest of the divisor, divide and manage the remainder as in simple numbers.*

EXAMPLES FOR PRACTICE.

3. Divide 16 cwt. 52 lb. by 4 cwt. 13 lb. Ans. 4.

4. Divide 25 lb. 10 oz. by 5 lb. 2 oz.

5. Divide 72 A. 96 P. by 12 A. 16 P. Ans. 6.

6. How many barrels, each holding 40 gal. 1 qt. 1 pt. 2 gi., can be filled from 363 gal. 3 qt. 1 pt. 2 gi.? Ans. 9.

7. A man, having 35 A. 1 R. 16 P. of land, laid it off into lots each containing 4 A. 1 R. 27 P. How many lots were there?

8. How many 4-ounce bottles (4f℥) can be filled from a cask of castor oil containing 8 gal. 3 qt. .75 pt.? Ans. 283.

9. How many 2-ounce bottles can be filled from 2 gal. 3 qt. 1 pt. of laudanum?

10. How many sacks, each holding 2 bu. 1 pk. 5 qt. 1 pt., can be filled from 104 bu. 4 qt. 1 pt. of grass seed? Ans. 43.

11. Divide 2 ℔ 7 ℥ 5 ʒ by 2 ℈ 8 gr. Ans. 316.25.

12. Divide 5 lb. 6 oz. 17 pwt. 10 gr. by 6 oz. 8 pwt. 16 gr.

13. How many bales of hay each weighing 5 cwt. 91 lb. are there in 9 T. 9 cwt. 12 lb.? Ans. 32.

14. Divide 35 gal. 1 qt. 1 pt. 2 gi. by 2 gal. 3 qt. 1 pt. 2.5 gi.

15. Divide £346 18s. 4d. 2 far. by £7 7s. 7d. 2 far. Ans. 47.

16. Divide 174 mi. 26 rd. by 12 mi. 3 fur. 19 rd. Ans. 14.

17. Divide the sum of 18 da. 5 hr. and 7 da. 22 hr. by the difference between 16 da. 4 hr. and 12 da. 10 hr. Ans. $6\frac{29}{30}$.

CASE II.

Art. 284. To divide a compound number by a simple whole number.

Ex. 1. Find $\frac{1}{5}$ of 144 A. 2 R. 10 P. Ans. 28 A. 3 R. 26 P.

WRITTEN PROCESS.

5)	144 A.	2 R.	10 P.
	28	3	26

EXPLANATION.

One-fifth of 144 A. is 28 A., and 4 A. remain undivided. This, being reduced to roods, is 16 roods, which with the 2 R. given makes 18 R. One-fifth of 18 R. is 3 R., and 3 R. remain undivided. This equals 120 P., which with the 10 P. given makes 130 P. One-fifth of 130 P. is 26 P.

Ex. 2. Find $\frac{1}{37}$ of £117 6s. 9d.

BY LONG DIVISION.

37)£117 16s. 9d. (£3.
111
6
20
37)136s. in £6 16s. (3.
111
25
12
37)309d. in 25s. 9d. (8
296
13
4
37)52 far. ($1\frac{15}{37}$ far.
37
15

BY SHORT DIVISION.

37)	£117	16s.	9d.	0 far.
	3	3	8	$1\frac{15}{37}$

REMARK.—When the divisor is large, and a prime number, it is often convenient to conduct the divisions and reductions in *full long division*, that every step may be presented to the eye.

Rule.—*Divide the numbers in the denominations in order from the highest to the lowest.*

If, in dividing the number in any denomination, there is a remainder, reduce it to the next lower denomination, add in the given number of this lower denomination, and divide the sum.

EXAMPLES FOR PRACTICE.

(3.)

	T.	cwt.	lb.	oz.
8)	102	2	20	8
	12	15	27	9

(4.)

	mi.	fur.	rd.	ft.
9)	133	6	11	10.5
	14	6	36	14

5. Divide 12 cords 5 cu. ft. 72 cu. in. by 5.
Ans. 2 cords 52 cu. ft. 360 cu. in.

6. Divide 13 ℔. 4 ℥ 4 ʒ 1 ℈ 10 gr. by 7.
Ans. 1 ℔. 10 ℥ 7 ʒ 1 ℈ 10 gr.

7. A contractor paved 29 mi. 4 fur. 23 rd. 3 ft. 10 in. of street in 5 years; how much did he average a year?

Ans. 5 mi. 7 fur. 12 rd. 10 ft. 8 in.

8. Divide 17 A. 2 R. 31 P. 3 ft. 20 in. by 2.

Ans. 8 A. 3 R. 15 P. 137 ft. 100 in.

9. Divide 37 gal. 3 qt. 1 pt. 3 gi. by 8.

10. Divide 157 yd. 2 ft. 4 in. by 32.

Ans. 4 yd. 2 ft. $9\frac{1}{2}$ in.

11. If a train of cars moves 348 mi. 1 fur. 12 rd. in 14 hours, how far does it move in 1 hour?

Ans. 12 mi. 3 fur. 19 rd.

12. A man purchased 322 A. 2 R. 10 P. of land, which he wished to divide into 26 equal fields; how much land will each field contain? Ans. 12 A. 1 R. 25 P.

13. Divide 1 cong. 5 O. 4 f℥ 4 fʒ by 50. Ans. 4 f℥ 2 fʒ.

14. Divide 306 yd. 2 ft. 9.79 in. by 83.

Ans. 3 yd. 2 ft. 1.13 in.

15. Divide 32 bu. 1 pk. 7 qt. 1.3465 pt. by 10.8724.

Ans. 2 bu. 3 pk. 7 qt. 1.25 pt.

16. Divide $2\frac{6}{7}$ mi. by 4. Ans. 5 fur. 28 rd. 9 ft. $5\frac{1}{7}$ in.

17. Divide 8° 16′ 20″ by $\frac{5}{6}$. Ans. 9° 55′ 36″.

NOTE.—Multiply by 6, the denominator, and divide the product by 5, the numerator, or divide by the numerator and multiply the quotient by the denominator.

18. Divide £17 6s. 4d. 2 far. by $2\frac{1}{2}$. (Multiply dividend and divisor by 2.)

19. Divide 41 yd. 2 qr. 3 na. 1 in. by $6\frac{2}{3}$.

Ans. 6 yds. 1 qr. $\frac{21}{80}$ in.

20. If a comet moves through 121° 36′ 30″ in 93 days, how much is that per day?

Art. 285. To find the difference of the time of two places, from their difference of longitude.

Ex. 1. If St. Petersburg is 30° 18′ 22.2″ E. from Greenwich, and Washington is 77° 2′ 48″ W. from Greenwich, what is the difference of the time of these places?

Ans. 7 hr. 9 min. 24.68 sec.

WRITTEN PROCESS.

$$
\begin{array}{rlll}
 & 30^\circ & 18' & 22.2'' \\
 & 77 & 2 & 48 \\
\hline
15) & 107^\circ & 21' & 10.2'' \text{ sum} = \text{diff. of long.} \\
\hline
 & 7 \text{ hr.} & 9 & 24.68 \text{ diff. of time.}
\end{array}
$$

EXPLANATION.

Since St. Petersburg is east of Greenwich, and Washington is west of Greenwich, the sum of their distances from Greenwich is their distance from each other. The difference of their time is, in units of time, $\frac{1}{15}$ of the difference of the units which express their difference of longitude. Hence the

Rule.—*Divide the difference of longitude by 15, and call the quotient of degrees* hours, *that of minutes of arc* minutes of time, *and that of seconds of arc* seconds of time.

EXAMPLES FOR PRACTICE.

2. Rochester, New York, is in longitude 77° 51′ W. and Galveston, Texas, is in longitude 94° 47′ 15″ W.; what is the difference of time between the two places?

Ans. 1 hr. 7 min. 45 sec.

3. If Pittsburgh is 79° 58′ 15″ W. from Greenwich, how much is Pittsburgh time behind Greenwich time?

Ans. 5 hr. 19 min. 53 sec.

4. Stockholm is 18° 3′ 30″ and Madras 80° 15′ 57″ E. from Greenwich; what is the difference of time?

Ans. 4 hr. 8 min. 49.8 sec.

5. St. Petersburg is 30° 19′ 45″ E. from Greenwich, and New Orleans is 89° 1′ 50″ W. from Greenwich; what is the difference of time? Ans. 7 hr. 57 min. $26\frac{1}{3}$ sec.

6. If Cincinnati is 84° 24′ W. from Greenwich, what time is it at the former when it is noon at the latter?

Ans. 6 o'clock 22 min. 24 sec. A. M.

7. What time is it at Greenwich when it is noon at Cincinnati? Ans. 5 o'clock 37 min. 36 sec. P. M.

8. If Constantinople is 28° 59′ E. from Greenwich, what time is it at Constantinople when it is 2 o'clock 15 min. 30 sec. P. M. at Greenwich?

Ans. 4 o'clock 11 min. 26 sec. P. M.

9. What time is it at Greenwich when it is 2 hr. 15 min. 30 sec. P. M. at Constantinople? Ans. 19 min. 34 sec. P. M.

10. If Baltimore is 76° 37′ W., what is the time at Greenwich when it is 10 o'clock A. M. at Baltimore?

Ans. 3 hr. 6 min. 28 sec. P. M.

11. What is the time at Baltimore when it is 10 o'clock A. M. at Greenwich?

12. Canton is 113° 14′ E., and Quebec 71° 12′ 15″ W. longitude; what is the difference in time?

13. St. Louis is 90° 15′ W. and Jerusalem 35° 32′ E. longitude; what time is it at St. Louis when it is 3 hr. 20 min. P. M. at Jerusalem? Ans. 6 hr. 56 min. 52 sec. A. M.

14. What time is it at Jerusalem when it is 6 hr. 25 min. 40 sec. P. M. at St. Louis?

Ans. 2 hr. 48 min. 48 sec. A. M. of the following day.

15. What time is it at St. Louis when it is 6 hr. 25 min. 40 sec. A. M. at Jerusalem?

Ans. 10 hr. 2 min. 32 sec. P. M. of preceding day.

16. The longitude of Chicago is 87° 30′ 30″ W., and of Cape of Good Hope 18° 28′ 45″ E.; when it is 6 hr. 30 min. A. M. at the former, what is the time at the latter?

17. When it is 7 hr. 30 min. A. M. at the latter, what is the time at the former?

18. If I start from Greenwich with true time, how must I change my watch on arriving in Chicago?

DUODECIMALS.

Art. 286. In measuring length, surface, and volume, calculations are sometimes made in feet and the duodecimal divisions of a foot. (See Art. 205, Note 5.)

Duodecimals are those fractions of a linear, square, or cubic foot, which are expressed in descending denominations, in each of which a unit is one-twelfth of the next higher.

Hence, whichever kind of foot is taken as the unit,

1 prime, $(1')$,	$=\frac{1}{12}$		of that foot.
1 second, $(1'')$,	$=\frac{1}{12}$ of $\frac{1}{12}$,	$=\frac{1}{144}$	" " "
1 third, $(1''')$,	$=\frac{1}{12}$ of $\frac{1}{144}$,	$=\frac{1}{1728}$	" " "
1 fourth, $(1'''')$,	$=\frac{1}{12}$ of $\frac{1}{1728}$,	$=\frac{1}{20736}$	" " "

Duodecimals are usually written as a compound number. Thus, 2 ft. 3′ 4″ 10‴ 11⁗. It is plain that, if the indices, (′, ″, ‴, &c.,) were omitted, and the parts of a foot were expressed as *a number in the duodecimal scale*, (see Art. 27,) we should have a mixed number, of which the *feet* would be units in the decimal scale, and the parts of a foot would be written after a separatrix, in the duodecimal scale. This would imply the use of two new symbols for *ten* and *eleven*. Thus, the above number might be written *ft. 2.34 t e.*

Art. 287. To find the value of a duodecimal expression in feet and a common or decimal fraction of a foot.

Ex. 1. Find the value of 8 ft. linear, 4′ 6″ 8‴.

Ans. $8\frac{41}{108} = 8.37\dot{9}6\dot{2}$.

WRITTEN PROCESS.

$$\begin{aligned}
8 \text{ ft.} &= & & 8 \text{ ft.}\\
4' &= \tfrac{4}{12} \text{ ft.} &=\ & \tfrac{576}{1728} \text{ ft.}\\
6'' &= \tfrac{6}{144} &=\ & \tfrac{72}{1728} \text{ ft.}\\
8''' &= & & \tfrac{8}{1728} \text{ ft.}\\
\hline
& & & 8\tfrac{656}{1728} \text{ ft.} = 8.37\dot{9}6\dot{2}.
\end{aligned}$$

Rule.—*Express the primes, seconds, &c., as common or decimal fractions, then add them, and annex their value to the number of feet.*

EXAMPLES FOR PRACTICE.

Find the value of

2. Eight ft. 5′ 6″ linear. Ans. $8\frac{11}{24}$.
3. Fifteen ft. 8′ 9″ 10‴ linear. Ans. $15\frac{547}{864}$.
4. Nine ft. 7′ 4′ 8‴ 3⁗ linear.

5. Three sq. ft. 3′ 5″ 7‴. Ans. $3\frac{499}{1728}$ sq. ft.

6. Four sq. ft. 2′ 3″ 4‴.

7. Six cu. ft. 5′ 6″ 8‴ 9⁗. Ans. $6\frac{3203}{6912}$ cu. ft.

8. Seven cu. ft. 4′ 3″ 2‴ 8⁗.

9. What will 54 sq. ft. 3′ 5″ 8‴ of blackboarding cost, at \$1.72$\frac{4}{5}$ per square foot? Ans. \$93.81$\frac{1}{5}$.

10. If 1 cu. ft. of ore weighs 518.4 lbs., how much would 4 cu. ft. 5′ 6″ 9‴ weigh? Ans. 1 T. 3 cwt. 10.9 lb.

Art. 288. Duodecimal expressions may be added, or subtracted, and multiplied or divided by a whole number, like compound numbers. (See examples in Compound Numbers.)

Art. 289. The product of *length* by *length* expresses *surface.* (See Art. 213.) Hence the product of

Linear feet by *linear feet* expresses *square feet;*

Linear feet by *linear primes* expresses twelfths of a square foot, that is, *primes in square measure;*

Linear primes by *linear primes* expresses twelfths of twelfths, or 144ths, or *seconds in square measure;* and the product of

Linear primes by *linear seconds* expresses 12ths of 144ths, that is, 1728ths, or *thirds in square measure.*

The product of *surface* by *length* expresses *volume.* (See Art. 218.) Hence the product of

Square feet by *linear feet* expresses *cubic feet;*

Square feet by *linear primes* expresses twelfths of a cubic foot, that is, *primes in cubic measure;*

Square primes by *linear primes* expresses 12ths of 12ths, or 144ths, or *seconds in cubic measure.* Hence the

Rule.—*The denomination of a product is expressed by the sum of the indices of its factors.*

EXAMPLES FOR PRACTICE.

Multiply	Multiply
1. 8 ft. by 7 ft. Ans. 56 sq. ft.	6. 4 sq. ft. by 5 ft. Ans. 20 cu. ft.
2. 5 ft. by 9 ft.	7. 6 sq. ft. by 4‴. Ans. 24‴ cu.
3. 8 ft. by 4′. Ans. 32′ sq.	8. 7″ sq. by 6″. Ans. 42⁗ cu.
4. 6 ft. by 6″. Ans. 36″ sq.	9. 5‴ sq. by 3′.
5. 7′ by 5″. Ans. 35‴ sq.	10. 6′ sq. by 8‴.

Art. 290. To multiply one duodecimal expression by another.

Ex. 1. Multiply 4 ft. 7′ linear by 3 ft. 5′ linear.

Ans. 15 sq. ft. 7′ 11″.

FIRST METHOD.	SECOND.
4 ft. 7′	4 . 7
3 ft. 5′	3 . 5
1 ft. 10′ 11″	1 . *t e*
13 ft. 9′	13 . 9
15 ft. 7′ 11″	15 . 7 *e*

EXPLANATION.

First, 5′ × 7′ = 35″ = 2′ 11″. Write the 11″ and carry the 2′ to the next product. Then, 5′ × 4 ft. = 20′: add in the 2′, making 22′ = 1 ft. 10′. Write the whole. Next, 3 ft. × 7′ = 21′ = 1 ft. 9′. Write the 9′ and carry the 1 foot to the next product. Lastly, 3 ft. × 4 ft. = 12 sq. ft. + 1 = 13 sq. ft.

NOTE.—The second method shows the numbers written, not as compound numbers usually are written, but as mixed decimals are written, namely, with a separatrix, and the fractional parts on its right expressed in the duodecimal scale. Then the multiplication and carrying are the same as in the first method, writing the symbol *e* for *eleven*, and *t* for *ten*. (See Art. 27.)

Rule.—*Write the denominations of the multiplier under like denominations of the multiplicand.*

Beginning at the right-hand, multiply in order the number in each denomination of the multiplicand by the number in each denomination of the multiplier, expressing each partial product by the sum of the indices of its factors. Reduce the products as in compound multiplication, and add the results by compound addition.

Examples for Practice.

2. How many square feet in a floor 5 ft. 14′ long, and 13 ft. 8′ wide? Ans. 209 sq. ft. 6′ 8″ = 209$\frac{5}{9}$ sq. ft.

3. How much granite in a block 4 ft. 8′ long, 3 ft. 7′ broad, and 2 ft. 5′ high?

4. How much outside surface has a box 4 ft. 5′ long, 3 ft. 7′ wide, and 2 ft. 11′ high? Ans. 62 sq. ft. 5′ 11″.

5. How much plastering in a ceiling 14 ft. 5′ by 12 ft. 7′?

6. What is the capacity of a bin 5 ft. 7′ long, 4 ft. 6′ wide, and 3 ft. 5′ high? Ans. 85 cu. ft. 10′ 1″ 6‴.

7. How much plastering in a room 20 ft. 5′ long, 14 ft. 6′ wide, and 10 ft. 7′ high, deducting a fire-place 4 ft. 5′ by 4 ft., and 3 windows, each 7 ft. 5′ by 3 ft. 7′?
Ans. 104 sq. yd. 1 sq. ft. 8′ 7″.

8. A man engaged to dig a cellar 17 ft. 4′ long, 13 ft. 5′ wide, and 5 ft. 7′ deep. After he has removed 6 ft. 6′ × 5 ft., 8′ × 3 ft. 10′, of earth, how much remains to be dug?

Art. 291. Since a dividend equals the product of its divisor and the quotient, the index of a dividend is the sum of the indices of its divisor and the quotient. Therefore the index of the quotient is the difference of the indices of the divisor and the dividend.

Examples for Practice.

Divide	Divide
1. 40 sq. ft. by 5 ft. Ans. 8 ft.	4. 27⁗ cu. by 3″. Ans. 9″ sq.
2. 12″ sq. by 3′.	5. 48″ sq. by 6′. Ans. 8′.
3. 56‴ sq. by 7′. Ans. 8″.	6. 42⁗ by 7′.

Art. 292. To divide one duodecimal expression by another.

Ex. 1. Area 20 sq. ft. 4′ 1″ 8‴: one side 5 ft. 6′ 7″; find the other. Ans. 3 ft. 8′.

WRITTEN PROCESS.

$$
\begin{array}{l}
5\text{ ft. }6'\ 7'')\ 20\text{ ft. }4'\ 1''\ 8'''\ (3\text{ ft. }8' \\
\underline{\quad 16 \quad\ 7\ \ 9 \quad} \\
\quad\ \ 3\text{ ft. }8'\ 4''\ 8''' \\
\underline{\quad\ \ 3 \quad\ \ 8\ \ 4\ \ 8 \quad}
\end{array}
$$

EXPLANATION.

Since area is the product of length by breadth, the quotient of area by length is breadth. In dividing we may proceed thus:—5 ft. in 20 ft. 4 times: on trying 4 times 5 ft. 6′ 7″ we find the product too great: therefore we try 3 times 5 ft. 6′ 7″, and produce 16 ft. 7′ 9″. Subtracting, we have a new dividend, 3 ft. 8′ 4″ 8‴ = 44′ 4″ 8‴. Now, 5 ft. in 44′ gives 8′. On finding the product of the divisor by 8′, we find it equal to the dividend, and the quotient is complete.

SECOND METHOD.

$$20\text{ ft. }4'\ 1''\ 8''' = 35156'''.$$
$$5\text{ ft. }6'\ 7'' = 799''.$$
$$35156''' \div 799'' = 44' = 3\text{ ft. }8'$$

EXPLANATION.

In this process the dividend is reduced to its lowest denomination, *thirds*, and the divisor to its lowest denomination, *seconds*, and the quotient is 44 *primes*, equal to 3 ft. 8′.

Rule.—*Divide as in Long Division, making and subtracting the partial products according to the processes of compound numbers;* or,

Reduce divisor and dividend each to the lowest denomination in it, then divide, and reduce the quotient to higher denominations, if necessary.

EXAMPLES FOR PRACTICE.

2. Divide 19 sq. ft. 6′ 8″ by 5 ft. 4′. Ans. 3 ft. 8′.

3. Divide 209 sq. ft. 6′ 8″ by 15 ft. 4′. Ans. 13 ft. 8′.

4. Divide 26 sq. ft. 6′ 11″ 5‴ 4⁗ by 7 ft. 2′ 4″.

5. The surface of a board 13 ft. 8′ long is 24 sq. ft. 2′ 5″. How wide is it?

6. If a pile of wood 3 ft. 6′ wide, and 4 ft. 3′ high, contains 3 cords 73 cu. ft. 4′ 10″ 6‴, how long is it?

Ans. 30 ft. 9′.

7. If a cellar 40 ft. 3′ long, and 7 ft. deep, contains 213 cu. yd. 24 cu. ft. 10′ 6″, how broad is it?

8. What must be the length of a room that is 21 ft. 6′ 8″, to contain 704 sq. ft. 8′ 11″ 6‴ 8⁗? Ans. 32 ft. 8′ 4″.

9. Divide 8 ft. 6′ 6″ by 3 ft. 5′.

10. If a box 3 ft. 3′ wide and 1 ft. 10′ high contains 44 cu. ft. 8′ 3″, what is its length? Ans. 7 ft. 6′.

11. If a pile of bark 24 ft. 6′ long, and 4 ft. 6′ wide, contains 707 cu. ft. 5′ 3″, how high is it? Ans. 6 ft. 5′.

Miscellaneous Examples for Practice.

1. In 32 yd. 3 qr. how many French ells? Ans. $21\frac{5}{6}$.

2. In 3 mi. 28 ch. 35 l. how many links?

3. Reduce $\frac{13}{45}$ of a rod to the fraction of a yard.

4. Reduce 3 ft. 2.5 in. to the decimal of a rod. Ans. $.19\dot{4}$.

5. If water is $8\frac{1}{2}$ fathoms deep, how many feet is that?

6. What is the height of a horse that is 16 hands high?

7. How many links in 45 ft. 10 in.? Ans. $69\frac{4}{9}$.

8. Find the value of .0571 of a mile.

9. What will be the cost of a tract of land 143 rods long and 63 rods wide, at $48.8 per acre? Ans. $2747.745.

10. At $$.27\frac{1}{2}$ per square yard, what will it cost to pave a sidewalk 60 ft. long and 10 ft. wide?

11. How many yards of carpeting 1 yard wide will be required to cover a floor 33.5 ft. long and $22\frac{1}{2}$ ft. wide? Ans. 83.75.

12. What will be the cost of plastering a room 36 ft. long, $26\frac{1}{2}$ ft. wide, and 15 ft. high, at 21 cents a sq. yd., making no deductions? Ans. $66.01.

13. What will be the cost of carpeting the same room with Brussels carpet $\frac{3}{4}$ of a yard wide, at $$2.62\frac{1}{2}$ per yard? Ans. $371.

14. What will be the cost of 8 boards, each 16 ft. long and 16 in. wide, at the rate of $36 per M.? Ans. $6.144.

15. The roof of a house is 36 feet long, and each side 18 ft. 6 in. wide; what will the roofing cost at $$4.33\frac{1}{3}$ a square? Ans. $57.72.

16. How many acres in a tract of land 35 ch. 75 l. long, and 24 ch. 24 l. wide? Ans. 86 A. 2 R. 25.28 P.

17. At $3.60 per square, what will be the cost of tinning both sides of a roof 36 ft. 6 in. in length, and each side 18 ft. 9 in. in width?

18. How many feet, board measure, in a log 15 in. square, and 18 ft. long? Ans. 337½.

19. What will be the cost of a plank 24 ft. long, 16 in. wide, and 7 in. thick, at the rate of $42 per thousand?

20. If a board be 15 in. broad, what length of it will make 12 square feet? Ans. 9⅗ ft.

21. At 62½ cents per cu. yd., what will be the cost of digging a cellar 15.5 ft. long, 12 ft. wide, and 5 ft. 4 in. deep? Ans. 22.96\frac{8}{27}$.

22. What will be the cost of building a stone wall 660 ft. long, 6 ft. high, and 2.5 ft. thick, at $6.87½ a perch?

23. A man bought a pile of bark 19.5 ft. long, 7¾ ft. wide, and 6 ft. 8 in. high, at $10.50 per cord; what did it cost him Ans. $82.646+.

24. What must be the length of a block of marble which is 2 ft. 9 in. wide, and 2.25 ft. high, that it may contain 40 cu. ft. 164.16 cu. in.? Ans. 6 ft. 5.76 in.

25. In a pile of wood 48 ft. long 4.4 ft. wide, and 6 ft. 4.8 in. high, how many cord feet?

26. Bought a barrel (31.5 gal.) of vinegar for $7.25, and sold it at 11 cts. a quart; how much did I gain? Ans. $6.61.

27. In 63 U. S. gallons how many Imperial gallons? Ans. 52.485+.

28. In 63 Imperial gallons how many U. S. gallons?

29. How many U. S. gallons will fill a cistern 4.5 ft. square, and 6 ft. deep? Ans. 908.883+.

30. At 3 cts. a pint how much cider can be bought for $10.50?

31. Reduce 97567 ♏ to higher denominations.

32. A druggist bought 1 ℔ 6 ℥ 4 ʒ 1.5 ℈ of quinine, at $2.80 an ounce, and dealt it out in doses of 10 gr. at 15 cts. each; how much did he gain? Ans. $81.675.

33. What will be the cost of 3 bu. 2 pk. 5.5 qt. of pecans at 5 cts. a pint?

34. A $4 a bushel what will 3 bu. 5 qt. $1\frac{1}{2}$ pt. chestnuts cost?

35. How many U. S. bushels will a bin contain that is 7.5 ft. long, $4\frac{1}{4}$ ft. wide, and 3.25 ft. deep? Ans. 83.244.

36. How many barrels of 40.6 gal. each in an oil tank 12. 5 ft. long, 10 ft. wide, and 6 ft. deep?

37. What will it cost to fill a bin 3 ft. 9 in. long, 3 ft. 5 in. wide, and 2 ft. 4 in. deep, with clover-seed, at $6.25 per bushel? Ans. $150.145+.

38. How much is 15 lb. 8 oz. 10 pwt. of gold worth, at $.05 a grain?

39. How much gold at $15.85 an ounce can be bought for $1581.88? Ans. 8 lb. 3 oz. 16 pwt. 1.5 gr.

40. In 13 *long tons* how many common tons?

41. A stationer bought a great gross of lead-pencils at $2.50 a gross, and sold them at $.31$\frac{1}{4}$ a dozen. What was his profit?

42. If 1 cu. ft. of water weighs 1000 oz., what is the weight of the water in a cistern 8.4 ft. long, 6 ft. 8 in. wide, and $4\frac{1}{2}$ ft. deep? Ans. 7 T. 17 cwt. 5 lb.

43. I bought in Ohio 500 bu. clover-seed at $5 a bushel, and paid $230 for shipping it to New Jersey, where I sold it at $6.40 a bushel; how much did I gain? Ans. $270.

44. John is 18. yr. 5 mo. 10 da. old, James 23 yr. 7 mo. 12 da., and Charles 28 yr. 6 mo. 11 da.; what is the average of their ages? Ans. 23 yr. 6 mo. 11 da.

45. If a druggist pay $35 a pound avoirdupois for 4 pounds of quinine, and sell it at $.18 a ℈, how much will he gain? Ans. $112.

46. Bought of A 83 A. 3 R. 15 P. of land, of B 3 times as much, less 16 A. 3 R. 12 P., and of C $\frac{1}{2}$ as much as of A and B; how much did I buy of B and C each, and how much in all? Ans. Of B 234 A. 2 R. 33 P., of C 159 A. 1 R. 4 P., and of all 477 A. 3 R. 12 P.

47. A merchant bought 6384 bu. of corn in Illinois, at \$.455 a bushel, and sold it in Pennsylvania at \$.56$\frac{1}{4}$ a bushel; how much did he gain, if the transportation cost him \$367.50? Ans. \$62.28.

48. How many gallons in a bin 6.5 ft. long, 5 ft. wide, and 4.25 ft. deep? How many bushels?

49. If a board be 15 in. broad, what length of it will make 15 sq. ft.? Ans 12 ft.

50. A farmer has three fields of wheat; from the first he obtains 228 bu. 2 pk. 1 qt.; from the second $\frac{1}{3}$ as much, + 56 bu. 1 pk. 4 qt. $\frac{3}{5}$ pt., and from the third, as much as from the other two, less 216.40625 bu.; how much did he obtain from the three fields? Ans. 505 bu. 3 pk. 1 qt.

51. In digging a cellar 24 ft. long and 10 ft. wide, 1860 cu. ft. of earth were removed; how deep is it?

52. From a tract of land containing 365 A. 2 R. 10 P., were sold 110 A. 2 R. 30 P., and the remainder was divided equally among 5 persons; how much had each?

Ans. 50 A. 3 R. 36 P.

53. Which contains the greatest number of acres, 6 miles square, or 6 square miles, and how much?

54. Reduce 347921″ to signs.

55. How many seconds in a sextant? Ans. 216000″.

56. If a watch is $\frac{3}{4}$ pure gold, how many carats fine is it?

57. If 1 cu. ft. of water weighs 1000 oz., what will be the weight of the water in a cistern 8.5 ft. long, 6 ft. 3 in. wide, and 3 ft. 9 in. deep? Ans. 6 T. 4 cwt. 51 lb. 2$\frac{3}{4}$ oz.

58. How many square feet of sheet-zinc will be required to line the bottom and sides of the same cistern?

Ans. 163.75 sq. ft.

59. How much wheat can be put into a barrel that will hold 36 gal. of molasses?

Ans. 3 bu. 3 pk. 3 qt. 1.5 pt. nearly.

60. What will be the cost of 15 score of lamp-chimneys at 75 cts. per dozen? Ans. \$18.75.

61. Reduce 7 cwt. 43 lb. 11 oz. 8 dr. to the decimal of a ton?

62. In gold 20 carats fine, what part is pure and what part is alloy?

63. Divide 512 A. 2 R. 30 P. of land among A, B, C, and D, so that A shall have $\frac{1}{5}$ of the whole — 8 A. 1 R. 20 P.; B $\frac{1}{4}$ of the remainder + 4 A. 3 R. 14 P.; C $\frac{1}{3}$ of the remainder + 15 A. 2 R. 24 P., and D the rest; how much will each receive? Ans. A 94 A. 26 P.; B 109 A. 1 R. 35 P.; C 118 A. 2 R. 27 P.; D. 190 A. 1 R. 22 P.

64. How many sections of water-pipe, each 15 ft. long, will be required to conduct water 1.71875 mi.?

65. A person has 76 gal. of wine, which he wishes to put into demijohns containing 2 gal. 1 qt. 1 pt. each; how many will be required? Ans. 32.

66. How much oats will it take to seed 87 acres, using 2 bu. 1 pk. 5 qt. to the acre?

67. When it is 10 o'clock A. M. at Stockholm, it is 3 hr. 24 min. 58 sec. A. M. at Wheeling; the longitude of Wheeling is 80° 42′ W.; what is the longitude of Stockholm?

Ans. 18° 3′ 30″ E.

68. Pittsburgh is in longitude 79° 58′ 15″ W., and Paris in longitude 2° 20′ 22″ E.; what time is it at Pittsburgh when it is 2 o'clock 15 min. A. M. at Paris?

Ans. 8 o'clock 47 min. $45\frac{8}{15}$ sec. P. M. the day before.

69. When it is 12 o'clock M. at Boston, it is 11 hr. 36 min. 18 sec. A. M. at Washington; what is the longitude of Boston, Washington being 77° 1′ W.?

70. If Rome is in longitude 12° 28′ 45″ E., and Detroit 82° 58′ W., what is the time at Rome when it is 10 hr. 20 min. P. M. at Detroit?

71. A note dated July 15, 1873, was made payable in 90 days; when was it due?

72. What is the exact number of days from Jan. 21, 1864, to Sept. 11, 1865?

73. How many seconds in a solar year?

74. How many bricks, without cement, will occupy the space of a cord?

75. How many bricks, with cement, will be required to build a wall 40 ft. long, 4.5 ft. high, and 13 in. thick?

76. *Pittsburgh, May 9, 1868.*

Jos. Graham & Co.

Bo't of W. McClintock, Jr. & Co.

May 1.	To *5 pcs. Lumber,*	*16 ft. 8×10*	@ *$35*	*per M.*		
June 8.	" *16* " "	*19* " *7× 9* "	*35*	" "		
June 12.	" *15* " "	*21* " *3× 4* "	*27*	" "		
Aug. 12.	" *17* " "	*18* " *6× 9* "	*28*	" "		
Aug. 15.	" *30000 Shingles,*	"	*6.75* "	"		
Sept. 8.	" *23250 Lath,*	"	*3.25* "	"		

$399.650⅙.

Rec'd Payment,

W. McClintock, Jr. & Co.

Per. W. R. Ford.

77. If an acre of land be divided into 18 building lots, each 120 ft. deep, what is their average width?

Ans. 20 ft. 2 in.

78. In travelling, when I arrived at Harrisburg, my watch, which was exactly right at the beginning of my journey, and a correct time-keeper, was 2 hr. 8 min. 37 sec. too slow; from what direction had I come, and how far?

79. A miller bought 7845 bu. of wheat which he ground; how many barrels of flour had he, provided each bushel yielded 39 lb. $3\frac{1}{5}$ oz.? Ans. 1569.

80. A and B together bought a web of silk; A paid for $\frac{5}{12}$ of it and B the remainder, and the difference between their shares is 5 yd. 2 qr. 3 na. 1.5 in.; what is each one's share?

Ans. A's 14 yd. 1 qr. 1 na. .375 in.; B's 20 yd. 1.875 in.

81. What cost 16.325 M. of cloth at $3.60 per M.?

Ans. $58.77.

82. If a book 4.5 c.m. thick has 400 leaves, what is the average thickness of the leaves? Ans. .1125 m.m.

83. Divide 11 d.m. by 5 m.m. Ans. 220.

84. What is the average of 3 K.M., 1.225 K.M., 200 M., 6.475 M., 7.5 d.m., 7.5 c.m., 7.5 D.M., 7.5 H.M.?

Ans. 657.1625 M.

85. Reduce 5 K.M. 7 D.M. 4 M. 2 c.m. to c.m.

Ans. 507402 c.m.

86. From 3.625 D.M. of cloth were cut 8.75 d.m.; how much remained? Ans. 35.375 M.

87. How many steps of 6.4 d.m. each will a man take in walking 4.88 M.M.? Ans. 76250.

88. A man who owned 1 K.M.^{2} of land, bought at one time 43.25 H.M.^{2}, at another 75 M.^{2}; how much had he then?

Ans. 1.432575 K.M.^{2}, or 1432575 M.^{2}

89. In 40 H.M.^{2} how many acres?

Ans. 98 A. 3 R. 14.4 P.

90. In 3.25 K.M.^{2}, 16.5 D.M.^{2}, 24 d.m.^{2}, 84 c.m.^{2}, how many M.^{2}? Ans. 3251650.2484 M.^{2}

91. How many square metres of plastering in a room 12.5 M. long, 6 M. wide, and 3.5 M. high, after deducting 2 doors each 1.75 M. by 8 d.m., and 3 windows each 1.5 M. by 6.4 d.m.? Ans. 200.74 M.^{2}

92. How many acres of land in a farm 7.25 H.M. long, and 6.4 H.M. wide? Ans. 114 A. 2 R. 24.704 P.

93. What will be the cost of digging a cellar 6.4 M. long, 5 M. wide, and 2.5 M. deep, at $.75 per M.^{3}? Ans. $60.

94. What will be the cost of a pile of wood 18.75 M. long, 1.2 M. wide, and 1.5 M. high, at $2.25 per S.?

Ans. 75.93\frac{3}{4}$.

95. In 2.5 M. × 8 M. × 6.25 M. how many cords?

Ans. 34.48828125.

96. A merchant bought 80 D.L. of wine at 30 cts. per L., and retailed it at 50 cts. per L. How much did he gain?

Ans. $160.

97. In 7.4213 H.L. how many m.l.?

98. In 3.25 L. how many gallons?

99. In 678253 c.l. how many K.L.? Ans. 6.78253 K.L.

100. How many bottles, each holding 8 d.l., can be filled from 4 cubic metres of liquid? Ans. 5000.

101. How many doses of medicine, averaging 4 m.l. each, are there in 1 L.? Ans. 250.

102. If alcohol costs $1.20 per l., what is the cost of 4.5 d.l.? Ans. 54 cts.

103. In 7.23 d.g. how many m.g.? Ans. 723 m.g.

104. Reduce 51678 G. to M.G. Ans. 5.1678 M.G.

105. In 7.423 M.G. how many G.? Ans. 74230 G.

106. In 3.16 D.G. how many d.g.?

107. If a 5-cent piece weighs 5 G., what is the weight of $800 of that coin? Ans. 8 M.G.

108. Reduce 25 H.G. to pounds. Ans. 5 lb. 8 oz. 2.96 dr.

109. How many metres of carpet 8 d.m. wide will be required to cover a floor 9 M. long, and 7.5 M. wide?

Ans. 84.375 M.

110. A man owns 3 fields whose dimensions in metres are 240 by 250, 280 by 225, and 320 by 162.5; how many hectares in all? How many acres?

Ans. 17.5 $H.M.^2$ 43 A. 38.8 P.

111. Reduce 100 M. to ft. Ans. $328\frac{1}{12}$ ft.

112. Reduce 100 ft. to metres. Ans. 30.47+ M.

113. Reduce 15 tonneaux to tons. Ans. 16.5345.

114. How many barrels of flour in 125 quintals?

Ans. 140.5 nearly.

115. How many square yards in 218 ares?

116. How many ares in 26072.8 sq. yds.?

117. How many bushels in 136425 d.l.?

118. How many H.L. in 350 bu. 1 pk. 1 qt. .52 pt.?

Ans. 123.45.

119. What is the average of 8.357 K.M.^{2}, 42.8 H.M.^{2}, 12.28 M.^{2}, 85 d.m.^{2}, and 25 c.m.^{2}? Ans. 1757002.6265 M.^{2}

120. What must be the width of a pile of wood, 8.12 M. long, and 1.52 M. high, to contain 16.04512 steres?

Ans. 1.3 M.

121. How long will a person be in travelling 1000 K.M. at the rate of 40 miles per day?

122. Reduce 37562 seconds to grades.

123. In 1 quadrant 73 grades 24 min., how many minutes?

124. In 12 qr. 17 min. 13 sec. how many seconds?

125. Find the difference between 5 mi., and 4 mi. 7 fur. 39 rd. 5 yd. 2 ft. 11 in.

126. A was born June 18, 1834, and died at the age of 27 yr. 5 mo. 20 da.; what was the date of his death?

SYNOPSIS OF OPERATIONS IN DENOMINATE NUMBERS.

OPERATIONS.	KINDS.	KINDS.	APPLICATION.	KINDS.
Reduction.	Integers. Fractions.	Ascending. Descending.		TIME. Chronology.
				POSITION. Lat. & Long.
Addition. Subtraction.	Integers. Fractions.	Simple. Compound.	Scales.	FORCE. Heat. Light. Pressure. Traction.
Multiplication. Division.	Integers. Fractions.	Simple. Compound.	Long. and Time.	

Reduction Addition Subtraction Multiplication Division	of Duodecimals.

CHAPTER XIV.

ANALYSIS.

Art. 293. Analysis, in Arithmetic, is a method of solving problems by separately computing each of their conditions. It is often called *practice*. The solving of a problem in this manner is called *analyzing* it. Properly speaking, all solutions are analyses, but the word *analysis* has often the above narrower meaning.

ANALYSIS BY ALIQUOT PARTS.

Art. 294. Analysis by aliquot parts is the process of computing separately the product of one number by the units in another, and by those portions which are aliquot parts of such units.

Ex. 1. What cost 27 yd. of flannel at $68\frac{3}{4}$ ct. per yd.?

Ans. $\$18.56\frac{1}{4}$.

FIRST METHOD.—$27 \times \$0.6875 = \$18.5625$.
SECOND METHOD.—$27 \times \$\frac{11}{16} = \$\frac{297}{16} = \$18.5625$.

PROCESS BY ALIQUOT PARTS.

50 ct. $= \$\frac{1}{2}$:	2	\$27.00	= cost at	\$1.00
$12\frac{1}{2}$ ct. $= \frac{1}{4}$:	4	13.50	= " "	\$0.50
$6\frac{1}{4}$ ct. $= \frac{1}{2}$:	2	3.375	= " "	$.12\frac{1}{2}$
		1.6875	= " "	$.06\frac{1}{4}$
		\$18.5625	= " "	$\$0.68\frac{3}{4}$

ANALYSIS.

At \$1 per yd. the cost of 27 yd. would be \$27. Since 50 ct. is $\frac{1}{2}$ of \$1, the cost at 50 ct. would be $\frac{1}{2}$ of \$27., or \$13.50. Since $12\frac{1}{2}$ ct. is $\frac{1}{4}$ of 50 ct., the cost at $12\frac{1}{2}$ ct. would be $\frac{1}{4}$ of \$13.50, or \$3.375. Since $6\frac{1}{4}$ ct. is $\frac{1}{2}$ of $12\frac{1}{2}$ ct., the cost at $6\frac{1}{4}$ ct. would be $\frac{1}{2}$ of \$3.375, or \$1.6875. Therefore the cost at 50 ct. + $12\frac{1}{2}$ ct. + $6\frac{1}{4}$ ct., or $68\frac{3}{4}$ ct., would be $\$13.50 + \$3.375 + \$1.6875 = \$18.5625$.

Ex. 2. If 1 gallon of alcohol weighs 6 lb. 14 oz. 15 dr., how much do 7 gal. 3 qt. 1 pt. 1 gi. of the same weigh?

Ans. 54 lb. 13 oz. 1.59375 dr.

PROCESS BY ALIQUOT PARTS.

2 qt. = $\frac{1}{2}$ gal. : 2	6 lb.	14 oz.	15 dr.	=	wt. of 1 gal.
			7		
	48 lb.	8 oz.	9 dr.	= "	7 gal.
1 qt. = $\frac{1}{2}$ of 2 qt. : 2	3	7	7.5	= "	2 qt.
1 pt. = $\frac{1}{2}$ of 1 qt. : 2	1	11	11.75	= "	1 qt.
1 gi. = $\frac{1}{4}$ of 1 pt. : 4	0	13	13.875	= "	1 pt.
	0	3	7.46875	= "	1 gi.
	54	13	1.59375	= "	7 gal. 3 qt. 1 pt. 1 gi.

Rule.—*Multiply by the units of the multiplier, and add to this product such aliquot parts of the multiplicand as the rest of the multiplier may be of its units.*

NOTE 1.—It is best to take each preceding aliquot part so that the following part may be found from it.

NOTE 2.—This method is rapid, and saves labor in problems like Ex. 2., which in other methods require much reduction.

EXAMPLES FOR PRACTICE.

What will be the cost

3. Of 72 lb. rice, at 12$\frac{1}{2}$ cts. per lb.? Ans. $9.
4. Of 42 yd. calico, at $.16$\frac{2}{3}$ per yd.?
5. Of 27 yd. broadcloth, at $3.50 per yd.? Ans. $94.50.
6. Of 24 bbl. flour, at 8.43\frac{3}{4}$ per bbl.? Ans. $202.50.
7. Of 84 bu. wheat, at 1.62\frac{1}{2}$ per bu.?
8. Of 48 bu. oats, at $.68$\frac{3}{4}$ per bu.?
9. Of 13 boxes cigars, at 2.16\frac{2}{3}$ per box?
10. Of 20 A. 3 R. 15 P. of land, at $36 per A.? Ans. $750.375.
11. Of 12 A. 3 R. 33 P. of land, at 35.68\frac{3}{4}$ per A.? Ans. $460.9683.
12. Of 49 A. 2 R. 39 P. of land, at 87.66\frac{2}{3}$ per A.?

13. Of 3 bu. 3 pk. 5 qt. 1 pt. of peas, at $4.75 per bu.? Ans. $18.62902625.

14. Of 11 lb. 13 oz. butter, at 31$\frac{1}{4}$ cts per lb.?

15. Of 3 lb. 10 oz. 8 pwt. 5.5 gr. of gold, at $15.46 per oz.? Ans. 717.521\frac{7}{48}$.

16. If a person travel 3 mi. 4 fur. 8 rd. in 1 hr., how far will he travel in 6 hr. 25 min. 40 sec.?

Ans. 22 mi. 5 fur. 10$\frac{8}{15}$ rd.

17. If the interest on a note for 1 yr. is $95.04, what will be the interest for 4 yr. 7 mo. 19 da.? Ans. $440.616.

18. Find the cost of 6 T. 13 cwt. 71 lb. of hay at $27 per ton. Ans. $180.5085.

19. If 231 cu. in. make 1 gallon, how many cu. in. in 7 gal. 3 qt. 1 pt.?

20. At the rate of 30 mi. per hour, how far will a train move in 3 hr. 10 min. 30 sec.? Ans. 95.25 mi.

21. If 5760 gr. make 1 lb. Troy, how many grains are there in 5 lb. 4 oz. 7 pwt. 12 gr.? Ans. 30900.

22. In 1 bu. there are 2150.42 cu. in.; how many cu. in. in 7 bu. 2 pk. 5 qt.?

23. If a man walk 3 mi. 3 fur. per hour, how far will he walk in 5 hr. 15 min. 20 sec.? Ans. 17 mi. 5 fur. 36 rd.

24. At $.87$\frac{1}{2}$ per pound, what will be the cost of 3 lb. 11 oz. of tea? Ans. $3.2265625.

25. If a man pay interest at the rate of $360 per year, how much interest will he pay in 2 yr. 11 mo. 19 da.?

Ans. $1069.

26. At $1.25 a cu. yd., what will it cost to remove 13 cu. yd. 21 cu. ft. of earth? Ans. 17.22\frac{2}{9}$.

27. What will be the cost of 9 A. 3 rd. 37 P. of land, at £6 15s. 11d. per acre? Ans. £67 15s. 7d. 2$\frac{23}{40}$ far.

28. In 1 ℔. Apothecaries' Weight there are 5760 gr.; how many in 3 ℔ 5 ℥ 5 ʒ 2 ℈?

29. What will 4 gal. 3 qt. 1 pt. of oil cost, at 1.37\frac{1}{2}$?

30. If the interest on a note for 1 yr. be $25.20, what will be the interest on the same note from Feb. 5, 1867, to Aug. *20, 1869?* Ans. $64.05.

GENERAL ANALYSIS.

Art. 295. Analysis, generally, proceeds according to the following

Rule.—*From the quantity depending upon a given number of units, or parts, infer the quantity depending upon one of those units or parts; and then, if necessary, from this quantity infer the quantity depending upon the required number.*

Questions to be Analyzed.

1. If 4 hats cost \$12, what cost 7 hats? Ans. \$21.

Analysis.—If 4 hats cost \$12, 1 hat will cost $\frac{1}{4}$ of \$12, which is \$3, and 7 hats will cost 7 times \$3, which is \$21.

2. If 13 lb. of rice cost \$1.92, what will be the cost of 20 lb.?

3. If 16 barrels of pork contain 3200 lb., how much do 25 barrels contain?

4. If 24 hours contain 1440 min., how many minutes in 15 hours?

5. If 10 A. of western land cost \$215, what will 23 A. cost?

6. If 11 reams of paper contain 5280 sheets, how many sheets in 9 reams?

7. If 12 barrels of flour contain 2352 lb., now much do 9 barrels contain?

8. If $\frac{7}{8}$ of an acre of land cost \$35, what will $\frac{4}{5}$ of an acre cost? Ans. \$32.

Analysis.—If $\frac{7}{8}$ of an acre cost \$35, $\frac{1}{8}$ of an acre will cost $\frac{1}{7}$ of \$35, which is \$5; and $\frac{8}{8}$, or 1 acre, will cost 8 times \$5, which is \$40. Then, since 1 acre costs \$40, $\frac{1}{5}$ of an acre will cost $\frac{1}{5}$ of \$40, which is \$8, and $\frac{4}{5}$ of an acre will cost 4 times \$8, which is \$32.

9. If $\frac{3}{5}$ of a man's age is 27 years, what is $\frac{7}{9}$ of his age?

10. If $\frac{4}{5}$ of a man's salary is \$1200, what is $\frac{5}{8}$ of it?

11. If $\frac{3}{7}$ of a ton of hay cost \$15, what will $\frac{5}{8}$ of $\frac{5}{8}$ of a ton cost?

12. If $1\frac{1}{2}$ yards of cloth cost \$1.20, what will $2\frac{3}{4}$ yards cost?

13. If $3\frac{2}{3}$ gallons of alcohol cost \$4.84, what will $1\frac{5}{8}$ gallons cost?

14. If $4\frac{3}{8}$ barrels of potatoes cost \$12.95, what will be the cost of $2\frac{1}{4}$ barrels?

15. If a man receive \$68.75 for $2\frac{3}{4}$ months' work, how much will he receive for $5\frac{2}{5}$ months?

16. If $5\frac{11}{20}$ T. of copper cost $2775, what would be the cost of $6\frac{9}{10}$ T. at the same rate?

17. If 3 men can paint a house in 14 days, how long should it take 7 men? Ans. 6 days.

ANALYSIS.—If 3 men can paint a house in 14 days, 1 man can paint it in 3 times 14 days, which is 42 days. If 1 man can paint it in 42 days, 7 men can paint it in $\frac{1}{7}$ of 42 days, which is 6 days.

18. If a vessel has provisions enough to last 24 sailors 40 days, how long would it last a crew of 30 men?

19. If 150 bushels of oats would last 6 horses 36 days, how long would it last, if 2 were added? Ans. 27 days.

20. If 8 men can build a wall in 24 days, how long should it take 12 men?

21. If 50 men can grade 3250 ft. of street in 48 days, how long should it take 75 men? Ans. 32 days.

22. If $800 gain $56 in 12 months, how long would $600 be in gaining the same amount?

23. If 600 bu. of coal keep 4 fires 8 months, how long should it keep 8 such fires?

24. If 3 men can mow 1 acre in 2 hr. 30 min., how long would it take 5 men, working at the same rate?

Ans. 1 hr. 30 min.

25. If an army of 12000 men has provisions sufficient to last 18 days, how long should the same provisions last, if 4000 men were discharged? Ans. 27 days.

26. If 11 men can build a house in 14 days, how long should it take 7 men?

27. If 6 men reap a field of 20 A. in $7\frac{1}{2}$ days how long should it take 9 men?

Art. 296. The foregoing forms of arithmetical analysis are useful in such problems as are discussed in the next chapter. Higher forms are used in problems whose conditions are not immediately proportional. Such problems are often solved by Algebra, which is a kind of General Arithmetic, but they are also analyzed in mental and written arithmetics, by processes which are substantially the same as those in Algebra.

CHAPTER XV.

RATIO. PROPORTION.

RATIO.

Art. 297. **Ratio** is the relation, in respect to value, which one quantity has to another of the same kind.

A ratio is a number which expresses ratio.

Ratio implies the comparison of one number with another, in respect to their value. Unlike numbers can have no ratio.

Like numbers may be compared, in respect to their value, in two ways, namely:—

FIRST.—By subtraction, to find *how much* one number *exceeds* the other. The difference is sometimes called the *arithmetical ratio*, but the name *difference* is generally preferred.

SECOND.—By division, to find *how many times* one number *contains* the other. The quotient is sometimes called the *geometrical ratio*, but the name *ratio* is generally preferred.

Art. 298. In estimating ratio, many mathematicians find the value of the first number as compared with the second. Thus, they consider the question "What is the ratio of 2 to 10?" as meaning "What is 2 as compared with 10?" and answer "Two is $\frac{1}{5}$ of 10, and the ratio of 2 to 10 is $\frac{1}{5}$." A few mathematicians find the value of the second number as compared with the first. To them the question "What is the ratio of 2 to 10?" means "What is 10 as compared with 2?" and they answer "Ten is 5 times 2, and the ratio of 2 to 10 is 5." In this work ratio will be considered as a comparison of the first number with the second, in accordance with the custom of nearly all French, German, and English writers.

Illustrations of Ratio.

1. What relation has 3 to 6? Ans. It is $\frac{3}{6}$, or $\frac{1}{2}$ as much.
2. What relation has 6 to 3? Ans. It is $\frac{6}{3}$, or twice as much.
3. What is 4, compared with 12? Ans. It is $\frac{4}{12}$, or $\frac{1}{3}$ as much.
4. What is 12, compared with 4? Ans. $\frac{12}{4}$, or 3 times as much.

Art. 299. The **terms** of a ratio are the two numbers which are compared.

The **antecedent** is the first term of a ratio.

The **consequent** is the second term of a ratio.

When both terms are mentioned together, they are sometimes called a **couplet**.

Art. 300. Ratio is indicated in two ways, namely:—

First.—By a fraction whose numerator is the antecedent, and whose denominator is the consequent.

Second.—By writing the consequent after the antecedent, with a colon between them.

Thus, the ratio of 3 to 7 is written either $\frac{3}{7}$, or 3 : 7, and is read "*three is to seven.*"

The value of a ratio is the quotient of the antecedent divided by the consequent.

Note.—Those who take the second view of ratio mentioned in Art. 298, write the consequent for the numerator, and the antecedent for the denominator, when writing ratio as a fraction; but, when using the colon, they write the antecedent before and the consequent after the colon.

Art. 301. The terms of a ratio must express not only the same kind of quantity, but the same *denomination*, if either is denominate. Thus, the ratio of 4 *inches* to 5 *feet* is not $\frac{4}{5}$; but, if we reduce 5 ft. to 60 in., the ratio is $\frac{4}{60} = \frac{1}{15}$.

Art. 302. Since the antecedent of a ratio is the *numerator of a fraction*, and the consequent is the denominator,

FIRST.—*The value of a ratio varies directly as the antecedent.* Therefore, multiplying the antecedent by a number multiplies the value of the ratio by that number; and dividing the antecedent by a number divides the value of a ratio by that number.

SECONDLY.—*The value of a ratio varies inversely as a consequent.* Therefore, multiplying the consequent by a number divides the value of the ratio by that number; and dividing the consequent by a number multiplies the value of the ratio by that number.

THIRDLY.—*The value of a ratio is constant when the terms vary proportionally.* Therefore, multiplying both terms by the same number, or dividing both terms by the same number, does not affect the value of the ratio.

EXERCISES.

1. Illustrate the first principle with $\frac{12}{60}$, $\frac{8}{32}$, $\frac{9}{54}$, $\frac{16}{112}$, $\frac{24}{48}$.
2. Illustrate the second principle with $\frac{10}{20}$, $\frac{7}{28}$, $\frac{4}{18}$, $\frac{5}{30}$, $\frac{21}{72}$.
3. Illustrate the third principle with $\frac{8}{12}$, $\frac{10}{20}$, $\frac{12}{36}$, $\frac{16}{64}$, $\frac{54}{81}$.

Art. 303. In reference to the number of its terms, a ratio is either *simple* or *compound.*

A **simple ratio** is a ratio which has only one antecedent and one consequent. Thus $3:4$, or $\frac{3}{4}$, is a simple ratio.

A **compound ratio** is a ratio which expresses the product of two or more ratios. Thus, if the ratio $3:4$ be compounded with the ratio $5:6$, the compound ratio is $3 \times 5 : 4 \times 6, = \frac{3}{4} \times \frac{5}{6} = \frac{5}{8}$.

A ratio which is compounded of two equal ratios is said to be *duplicate* of either of them. Thus, the ratio compounded of $2:4$ and $3:6$, each being equal to $\frac{1}{2}$, is equal to the ratio $1^2:2^2$, or $\frac{1}{4}$, and $1:4$ is the duplicate ratio of each constituent of the compound ratio. A ratio compounded of three equal ratios is said to be *triplicate* of the constituent ratios, and is the ratio of the cubes of its terms; a ratio

compounded of four equal ratios is said to be *quadruplicate* of the constituent ratios, and is the ratio of the fourth powers of its terms; &c. (See Art. 44.)

Art. 304. The ratio of the reciprocals of two numbers is called the **reciprocal ratio**, or **inverse ratio**, of those numbers. Thus, the reciprocal ratio of $3:4$, or $\frac{3}{4}$, is $\frac{1}{3}:\frac{1}{4}$, or $\frac{4}{3}$. The reciprocal ratio of two numbers is equal to the reciprocal of their ratio. Thus, the reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$.

The ratio of two numbers is sometimes called *direct*, to distinguish it from their inverse, or reciprocal ratio. Thus, the direct ratio of 3 to 4 is $3:4$, or $\frac{3}{4}$; and their reciprocal ratio is $4:3$, or $\frac{4}{3}$.

Art. 305. A **ratio of equality** is a ratio whose value is 1. Thus, $3:3$, $2\times 9:3\times 6$ are ratios of equality.

A **ratio of greater inequality** is a ratio whose value is greater than 1. Thus, $6:4$, $4\times 7:3\times 5$ are ratios of greater inequality.

A **ratio of less inequality** is a ratio whose value is less than 1. Thus, $5:7$, $2\times 3:5\times 8$ are ratios of less inequality.

Art. 306. *If the same number be added to both terms of a ratio of equality, the value of the ratio is not changed.* Thus, $2:2=2+1:2+1=3:3$, a ratio of equality.

Art. 307. *If the same number be added to both terms of a ratio of greater inequality, the value of the ratio is diminished; and, if the same number be subtracted from both terms of a ratio of greater inequality, the value of the ratio is increased.* Thus, $16:8=2$; $16+4:8+4=20:12=1\frac{2}{3}$; and $16-4:8-4=12:4=3$.

Art. 308. *If the same number be added to both terms of a ratio of less inequality, the value of the ratio is increased; and, if the same number be subtracted from both terms of a ratio of less inequality, the value of the ratio is diminished.* Thus, $8:16=\frac{1}{2}:8+4:16+4=12:20=\frac{3}{5}$; and $8-4:16-4=4:12=\frac{1}{3}$.

NOTE.—The fact that an equal increase or decrease of the less of two numbers is more in proportion to the value of that number than the other, explains the facts of Articles 306 and 307.

Art. 309. *The ratio of the sum of the antecedents of two or more equal ratios to the sum of the consequents is the same as the ratio of any one of the antecedents to its consequent.* Thus, in each of the couplets $15:3$, $25:5$, and $30:6$, the ratio is 5; and in $15+25+30:3+5+6$, or $70:14$, the ratio is 5. Again, in $\frac{4}{2}$, $\frac{6}{3}$, and $\frac{8}{4}$, each ratio is 2; also, in $\frac{18}{9}$, (formed as described,) the ratio is 2.

Art. 310. *The ratio of the difference of the antecedents of two equal ratios to the difference of the consequents is the same as the ratio of either antecedent to its consequent.* Thus, in each couplet $45:5$, and $18:2$, the ratio is 9; and in $45-18:5-2$, or $27:3$, the ratio is 9. Again, in $\frac{4}{2}$ and $\frac{10}{5}$ each ratio is 2; also, in $\frac{6}{3}$, (formed as described,) the ratio is 2.

PROPORTION.

Art. 311. Proportion, in general, is similarity of the relations of quantities. Proportion in two magnitudes, two structures, two systems, or two mixtures, exists when the parts of one are related to each other and the whole just as the parts of the other are related to each other and the whole.

Proportion, in Arithmetic, is equality of ratios.

A proportion is a statement of the equality of ratios. Thus, the equation $\frac{10}{5}=\frac{12}{6}$ is a proportion. A statement of the equality of the *value* of ratios is not always a proportion. Thus, in the above proportion, the equation, $2=2$, expressing the value of the ratios, is not a proportion. Every term of the ratios must be a term in a proportion formed by them.

Art. 312. The equality of one ratio to another is indicated in three ways, namely:—

FIRST.—As the equality of one fraction to another, as $\frac{8}{4}=\frac{6}{3}$.

SECOND.—By writing the ratios with colons, and placing the sign of equality between them; as $8:4=6:3$.

THIRD.—By writing the ratios with colons, and placing a double colon between them; as $8:4::6:3$.

This last is the common method, and the proportion given is read "*8 is to 4 as 6 is to 3.*"

Art. 313. Proportionals, or the **terms of a proportion,** are the numbers which form a proportion.

An **identical proportion** is a statement of the equality of a ratio to itself, or of two identical ratios. Thus, $\frac{8}{4} = \frac{8}{4}$, or $8 : 4 :: 8 : 4$ is one identical proportion. Such a proportion is as useless for information as an identical proposition, like "Iron is iron," "Coal is coal," &c.

Two numbers can form an identical proportion.

Three numbers are proportionals when the first is to the second as the second is to the third; thus, $3 : 6 :: 6 : 12$. In this case the second term is called a **mean proportional** between the other two numbers; also, the ratio of the first to the third is duplicate of that of the first to the second. Thus, $\frac{3}{6} = 2$, but $\frac{3}{12} = 4 = 2^2$.

Four numbers are proportionals when the first is to the second as the third is to the fourth: thus, $3 : 5 :: 6 : 10$.

Numbers are called **continual proportionals** when they can be arranged in such an order that the ratio of every two adjacent numbers is the same; that is, the first is to the second as the second is to the third, as the third is to the fourth, as the fourth is to the fifth, &c. Such numbers so arranged are said to be *in progression,* or to form *an equirational series.* Thus, 2, 4, 8, 16, 32, 64, &c., are arranged in progression, with a constant ratio 2.

Art. 314. *When four numbers are in proportion, the ratio of the first to the third is the same as that of the second to the fourth.* Thus, in $2 : 4 :: 6 : 12$, we have $2 : 6 :: 4 : 12$. This is sometimes called *taking a proportion alternately,* or *by permutation.*

Art. 315. *When four numbers are in proportion, the ratio of the second to the first is the same as that of the fourth to the third.* Thus, in $2 : 4 :: 6 : 12$ we have $4 : 2 :: 12 : 6$. This is sometimes called *inversion,* or *taking a proportion inversely.*

Art. 316. *When four numbers are in proportion, the sum of the first and second has the same ratio to the second that the sum of the third and fourth has to the fourth.* Thus, in $2:4::6:12$ we have $2+4:4::6+12:12$, or $6:4:18:12$. In like manner, $2+4:2::6+12:6$, or $6:2::18:6$. These operations are called *taking a proportion by composition.*

Art. 317. *When four numbers are in proportion, the difference of the first and second is to the second as the difference of the third and fourth is to the fourth.* Thus, in $2:4::6:12$ we have $4-2:4::12-6:12$, or $2:4::6:12$. In like manner, $4-2:2::12-6:6$, or $2:2::6:6$. These operations are called *taking a proportion by conversion.*

Art. 318. In a proportion of four terms,

The **extremes** are the first and fourth terms.

The **means** are the second and third terms.

Homologous terms in any number of proportionals are those of the same name. Thus, the antecedents are homologous to each other, and the consequents are homologous to each other.

Thus, in the proportion $2:4::6:12$, the extremes are 2 and 12; the means are 4 and 6; the antecedents 2 and 6 are homologous, and the consequents 4 and 12 are homologous.

Art. 319. In reference to the number of terms in their ratios, proportions which are composed of two ratios are *simple* or *compound.*

A **simple proportion** is a statement of the equality of two simple ratios. Thus, $3:4::6:8$ is a simple proportion.

A **compound proportion** is a statement of the equality of a simple to a compound ratio, or of one compound ratio to another. Thus, $3\times4:5\times6::2:5$ is a compound proportion, because it states that the ratio of 3 to 5, compounded with that of 4 to 6, equals that of 2 to 5. Also, $3\times5:4\times6::6\times10:8\times12$ is a compound proportion, because it states that the ratio of 3 to 4, compounded with that of 5 to 6, equals that of 6 to 8, compounded with that of 10 to 12, or that $\frac{3}{8}\times\frac{5}{4}=\frac{6}{12}\times\frac{10}{8}$.

Art. 320. *If, in a proportion of two ratios, both terms of a ratio are multiplied or divided by the same number, the resulting numbers are proportional with the other numbers.* This follows from the third corollary of Art 302.

Thus, the proportion $12 : 16 :: 36 : 48$ may become

By multiplication, $24 : 32 :: 36 : 48$, or $12 : 16 :: 72 : 96$, &c.

By division, $6 : 8 :: 36 : 48$, or $12 : 16 :: 18 : 24$, &c.

and each of these results is a proportion.

Art. 321. *If, in a proportion of two ratios, homologous terms are multiplied or divided by the same number, the resulting numbers are proportional with the other numbers.* This is because both sides of the equation are equally affected by the multiplier or the divisor. (Axioms 3 and 4, Art. 13.) Thus, $\frac{4}{8} = \frac{6}{12}$ may become $\frac{16}{8} = \frac{24}{12}$, or $\frac{2}{8} = \frac{3}{12}$, or $\frac{4}{32} = \frac{6}{48}$, or $\frac{4}{2} = \frac{6}{3}$, &c., and each of these results is a proportion.

Art. 322. *The products of the corresponding terms of two sets of proportionals are proportional.* This is because both sides of the equation are multiplied by equal quantities.

Thus, $\frac{2}{4} = \frac{6}{12}$, or $2 : 4 :: 6 : 12$	Also, $2 : 4 :: 6 : 12$
$\frac{3}{5} = \frac{9}{15}$, or $3 : 5 :: 9 : 15$	$2 : 4 :: 6 : 12$
$\frac{6}{20} = \frac{54}{180}$, or $6 : 20 :: 54 : 180$	$2^2 : 4^2 :: 6^2 : 12^2$

COROLLARY 1.—*Similar powers of proportionals are proportional.*

COROLLARY 2.—A compound proportion is equivalent to a simple proportion, whose terms are the product of all the corresponding terms of the compound proportion. (See Art. 319.)

COROLLARY 3.—The quotients of corresponding terms are proportional

Art. 323. *The product of the extremes of a proportion is equal to the product of its means.*

DEMONSTRATION.—Let a represent the first antecedent, c its consequent, and a' the second antecedent, and c' its consequent. Let n represent the number of times that a contains c. Then, for $a : c :: a' : c'$ we can put $n \times c : c :: n \times c' : c'$ The product of these extremes, $n \times c \times c'$, equals the product of the means, $n \times c \times c'$, whatever the value of n may be.

Art. 324. *If three terms of a proportion are given, the fourth may be found, if it is an extreme, by dividing the product of the means by the given extreme; or, if it is a mean, by dividing the product of the extremes by the given mean.* This is because the given extreme, or mean, is one factor of a product equal to the product of the other terms.

Illustrations.

1. In the defective proportion 2 : 4 :: 5 : what? We have $4 \times 5 = 2 \times$ the answer. Therefore the answer $= (4 \times 5 = 20,) \div 2 = 10$.

2. 2 : 4 :: what? : 10. Ans. $(2 \times 10$, or $20,) \div 4 = 5$.
3. 2 : what? :: 5 : 10. Ans. $(2 \times 10$, or $20,) \div 5 = 4$.
4. What? : 4 :: 5 : 10. Ans. $(4 \times 5$, or $20,) \div 10 = 2$.

Exercises.

Find the unknown term, (represented by ?,) of

1. 5 : 8 :: 10 : ?.
2. 6 : 9 :: ? : 12.
3. 8 : ? :: 20 : 100.
4. ? : 12 :: 8 : 16.
5. 5 : 3 :: $6\frac{2}{3}$: ?.
6. 3.5 : 17.5 :: ? : 16.
7. ? : $2\frac{1}{3}$ :: $3\frac{1}{4}$: $1\frac{1}{5}$.
8. \$6 : \$$1\frac{1}{2}$ :: $2\frac{3}{4}$ T. : ? T.
9. $1\frac{1}{2}$ lb. : 6 lb. :: \$? : \$$2\frac{3}{4}$.
10. 3.75 ft. : ? :: 7.5 in. : $8\frac{1}{3}$ in.
11. $\frac{5}{6}$: $\frac{3}{4}$ :: $\frac{7}{8}$: ?.
12. $\frac{1}{3}$: $\frac{3}{5}$ :: $1\frac{3}{5}$: ?.

SIMPLE PROPORTION.

Art. 325. Simple Proportion is employed to find a fourth proportional, when three are given.

Art. 326. In solving questions by simple proportion,

FIRST—The question must have the right number and kind of conditions. These are three numbers, two of which are *like numbers*, and a third, dependent on one of the two, and of the same kind as the number sought, which is dependent on the other of the two.

SECONDLY—The operator must state the conditions in their proper order. The two like numbers, on which the others

depend, must form a ratio, and the third must be one term of an unfinished ratio. In forming the complete ratio, regard must be had to whether the kind of quantity sought varies directly or inversely as that in the complete ratio.

THIRDLY—After a correct statement, the term sought is always found by the same method, namely, that stated in Art. 324.

Art. 327. To solve problems in Simple Proportion.

Ex. 1. If 16 yd. of cloth cost $18, how much would 22 yd. of the same cost? Ans. $24.75.

COMMON PROCESS BY PROPORTION.

$$\begin{array}{r} 16 \text{ yd.} : 22 \text{ yd.} :: \$18 : \text{cost of } 22 \text{ yd.} \\ \underline{22} \\ 16)\ \$396\ (\$24\tfrac{3}{4}. \end{array}$$

BY CANCELLATION.

$$\begin{array}{r} \cancel{16} \text{ yd.} : \cancel{22} \text{ yd.} :: \$18 : \text{Ans.} \\ \cancel{8} \text{ yd.} : 11 \text{ yd.} :: \$\cancel{18} : \text{Ans.} \\ 4 \text{ yd.} : 11 \text{ yd.} :: \$\ 9 : \text{Ans.} \\ \underline{11} \\ 4)\ \underline{\$99} \\ \$24\tfrac{3}{4}, \text{ Ans.} \end{array}$$

OTHER ARRANGEMENTS OF TERMS.

Cost of 22 yd. : $18 : : 22 yd. : 16 yd.
$18 : cost of 22 yd. : : 16 yd. : 22 yd.
22 yd. : 16 yd. : : cost of 22 yd. : $18.

EXPLANATION.

Since cost varies directly as quantity, 16 yd. must have to 22 yd. the same ratio that the cost of 16 yd., which is $18, has to the cost of 22 yd. Making the required cost the last extreme, it is found by multiplying together the two means, and dividing the product by the first extreme. Before doing this, we may cancel factors found in the first extreme and either mean, finding 4 yd. : 11 yd. : : $9 : Ans., as the least terms of the proportion. (See Arts. 319, 320.)

NOTE.—If either of the other arrangements is preferred as a method of statement, solve by Art. 324.

WRITTEN ANALYSIS BY RATIO.

$$\tfrac{22}{16} \times \$18 = \$24\tfrac{3}{4}.$$

ANALYSIS.—If 16 yd. cost $18, the cost of 22 yd. will be $\frac{22}{16}$ of $18, or $24¾.

COMMON ANALYSIS.—If 16 yd. cost $18, 1 yd. costs $\frac{1}{16}$ of $18, or $\$\frac{18}{16}$, and 22 yd. cost 22 times $\$\frac{18}{16}$, or $24¾.

Ex. 2. If a person can do a certain work in 12 days, by working 10 hours a day, in how many days can he do it by working 8 hours a day? Ans. 15.

COMMON PROCESS BY PROPORTION.

$$8 \text{ hr.} : 10 \text{ hr.} :: 12 \text{ da.} : ?$$
$$\underline{\quad 10}$$
$$8\,)\,120 \text{ da.}\,(\,15 \text{ da.}$$

EXPLANATION.

Since the number of days, necessary to do a certain work, varies inversely as their length, as the required days are shorter, more will be needed, and, if this greater number of days is the consequent of the second ratio, the greater number of hours, 10, must be the consequent of the first ratio: that is, the greater number of hours must have the same place in the ratio of hours, that the answer has in its ratio.

WRITTEN ANALYSIS BY RATIO.

$$\tfrac{10}{8} \times 12 \text{ da.} = 15 \text{ da.}$$

ANALYSIS.—If in 12 da. of 10 hr. each he can do the work, when the days are 8 hr. each it will require more days in the ratio of 10 to 8, that is, $\frac{10}{8}$ of 12 da., or 15 da.

COMMON ANALYSIS.—Working 10 hr. per da., in 12 da. he works 12 times 10 hr., or 120 hr. If he works only 8 hr. per day, he must, to do the same work, work as many days as 8 hr. is contained times in 120 hr., that is, 15 days.

Rule.—*Write, for the third term, that number which is of the same kind as the number sought.*

Of the other two numbers, write the greater for the second term when the answer should be greater than the third term; otherwise write the less for the second term.

Multiply the second and third terms together, and divide the product by the first term.

NOTE 1.—If the number sought is made any other term of the proportion than the fourth, each ratio must be constructed so that both are equal and represent the conditions of the question: then solve by Art. 324.

NOTE 2.—Before multiplying means or extremes together in solving any statement, the terms of a ratio should be in the same denomination, if they are of different denominations.

NOTE 3.—After arranging the terms, before multiplying and dividing, it is best to cancel all factors common to such term as will be the divisor and the other two terms.

NOTE 4.—If the given term of the unfinished ratio is compound, it may either be treated in the operation as a compound number, or be reduced to its lowest denomination and treated as a single number.

NOTE 5.—The learner should, for practice, solve the same question by both Analysis and Proportion. (See Chapter XIV.)

NOTE 6.—This rule is sometimes called "*The Single Rule of Three.*"

EXAMPLES FOR PRACTICE.

STATEMENT OF EXAMPLE 1 BY CAUSE AND EFFECT.

$$\left\{\begin{matrix}\text{1st cause}\\ \text{16 yards}\end{matrix}\right\} : \left\{\begin{matrix}\text{2d cause}\\ \text{22 yards}\end{matrix}\right\} :: \left\{\begin{matrix}\text{1st effect}\\ \$18\end{matrix}\right\} : \left\{\begin{matrix}\text{2d effect}\\ \$?\end{matrix}\right\}$$

NOTE.—Some prefer to consider the conditions of a question as causes and effects. Thus, in this statement, the quantity of cloth is considered the cause of the quantity of cost. It is plain that effects vary directly as their causes. This method should be used occasionally for variety.

3. If 15 bu. of wheat cost $24, what will 60 bu. cost?

4. If 36 men do a piece of work in 4 da., how long will 15 men be in doing the same? Ans. $9\frac{3}{5}$ da.

5. If 40 bu. of wheat make 8 bbl. of flour, how many bushels will be required to make 12 bbl.? Ans. 60.

6. If 90 bu. of wheat make 18 bbl. of flour, how many barrels of flour will 150 bu. make?

7. If a train, at 18 mi. per hour, go a certain distance in 8 hr., how long would it take it to go the same distance, at 30 mi. per hour?

8. If 25 men can dig a trench in 30 days, how long would it take 15 men to dig it?

9. If 15 men can build a wall in 50 days, how many men would build it in 30 days?

10. What will 18.75 yd. of cloth cost, if $1\frac{3}{4}$ yd. cost $5.60?

11. If $44 buy 16 hats, how many hats will $27.50 buy?

12. If $\frac{5}{8}$ of a yard of cloth cost $3.75, what will $\frac{4}{9}$ of a yard cost?

13. If 15 gal. 2 qt. 1 pt. of syrup cost $25, what will 20 gallons cost? Ans. $32.

14. If a tree 60 ft. high casts a shadow 80 ft. long, how high must a tree be that will cast a shadow 54 ft. long? Ans. 40.5 ft.

15. What is the cost of 15 lb. of butter, if 132 lb. cost $47.85?

16. If a pulse beats 24 times in 18 seconds, how many times a minute is that? Ans. 80.

17. What will $\frac{4}{5}$ of a barrel of sugar cost, if $6\frac{2}{3}$ barrels cost \$112.75?

18. If $\frac{5}{8}$ of a farm is worth \$$2408\frac{1}{3}$, what is $\frac{15}{17}$ of it worth?

19. If it require 45 yards of carpet, $\frac{7}{8}$ of a yard wide, to cover a floor, how many yards, $1\frac{3}{4}$ yards wide, will cover the same floor? Ans. $22\frac{1}{2}$.

20. If an 8-cent loaf weigh 9 ounces when flour is \$8 a barrel, what should it weigh when flour is \$10 a barrel? Ans. 7.2 oz.

21. If $5\frac{1}{2}$ bu. rye cost \$$7\frac{2}{3}$, what will $13\frac{3}{4}$ bu. cost? Ans. \$$19\frac{1}{6}$.

22. If a staff $6\frac{1}{2}$ feet long cast a shadow $8\frac{2}{3}$ feet long, how high is a steeple whose shadow at the same time is 275 feet long? Ans. $206\frac{1}{4}$ ft.

23. If \$170 gain \$10.20, what sum will gain \$600, in the same time, at the same rate? Ans. \$10000.

24. If 15 men can complete a piece of work in 24 days, how many men must be added to the number, that the work may be completed in $\frac{5}{6}$ of the time? Ans. 3.

25. If interest for 2 yr. 7 mo. is \$55.80, what would be the interest for 1 yr. 5 mo.?

26. If 7 barrels of flour are sufficient for a family 6 months, how many barrels will they require for 11 months?

27. If a grocer used a gallon measure that is deficient by 1.5 gi., how many of his *false* gallons would be in a barrel containing 39.65 gallons? Ans. 41.6.

28. If £1 sterling equals \$$4.86\frac{6}{10}$ U. S. money, how much do £5 6s. 10d. equal?

29. If a wheel containing 25 cogs works in another containing 9 cogs, how many revolutions will the smaller wheel make while the larger wheel makes 18 revolutions? Ans. 50.

30. If A who is worth \$2880, is taxed \$19.20, what is B worth who is taxed \$128?

31. If $\frac{2}{3}$ of an apple cost $\frac{3}{5}$ of a cent, what is $\frac{5}{6}$ of an apple worth? Ans. $\frac{3}{4}$ of a cent.

32. If $\frac{5}{9}$ of a ship is worth \$250000, what is $\frac{7}{15}$ of it worth?

COMPOUND PROPORTION.

Art. 328. Compound Proportion is employed when the solution of a question by Simple Proportion requires more than one statement. In such questions the answer depends on two or more pairs of conditions.

Art. 329. To solve problems in Compound Proportion.

Ex. 1. If 48 men, working 24 days of 10 hours each, earn \$1728, what, at the same rates, would 60 men earn, working 36 days of 8 hours each? Ans. \$2592.

SOLUTION BY ANALYSIS.—Since 48 men, working 24 times 10 hours, or 240 hours, earn \$1728, the average earning of 1 man in 240 hours is $\frac{1}{48}$ of \$1728, or \$36; and in 1 hour it is $\frac{1}{240}$ of \$36, or 15 cents. Therefore, at the same rate, 60 men would earn in 1 hour 60 times 15 cents, or \$9, and in 36 times 8 hours, or 288 hours, they would earn 288 times \$9, or \$2592.

SOLUTION BY SIMPLE PROPORTION.

FIRST PROPORTION.—48 men : 60 men : : \$1728 : \$2160.
SECOND PROPORTION.—24 days : 36 days : : \$2160 : \$3240.
THIRD PROPORTION.—10hours : 8 hours : : \$3240 : \$2592.

EXPLANATION.—The first proportion shows that, if 48 men earn \$1728, 60 men would, on the same conditions, earn \$2160. The second proportion shows that, if in 24 days 60 men earn \$2160, they would in 36 days earn \$3240. The third proportion shows that if, working 10 hours a day, 60 men earn \$3240, they would, working 8 hours a day, earn \$2592.

SOLUTION BY COMPOUND PROPORTION.

$$\begin{array}{l} 48 \text{ men} : 60 \text{ men} : : \$1728. \\ 24 \text{ days} : 36 \text{ days} \\ \underline{10 \text{ hours} : 8 \text{ hours}} \\ \text{Prod. } 11520 : \text{Prod. } 17280 : : \$1728 : \$2592. \end{array}$$

BY CANCELLATION.

$$\frac{\overset{\cancel{6}}{\cancel{60}} \times \cancel{36}^{3} \times \cancel{8} \times \$1728}{\cancel{48} \times \cancel{24}_{2} \times \cancel{10}} = \frac{\$5184}{2} = \$2592$$

EXPLANATION.

Since dollars are the required term, we assign for its place the consequent of the second ratio, and make the given number of dollars, \$1728, its antecedent. Now, separately considering each pair of the remaining conditions, since the demanded 60 men will earn more than the stated 48 men, make

60, the greater number of men, a consequent, and 48, the less number of men, an antecedent of the first ratio. Again, since the demanded 36 days will earn more than the stated 24 days, make the greater number of days, 36, a consequent, and the less number of days, 24, an antecedent of the first ratio. Again, since the demanded 8 hours will earn less than the stated 10 hours, make the less number of hours, 8, a consequent, and the greater number of hours, 10, an antecedent of the first ratio. The first ratio is, therefore, compounded of 48 : 60, of 24 : 36, and of 10 : 8, whose product forms the simple proportion 11520 : 17280 :: $1728 : $2592. But, cancelling factors common to any first term and any of the others, the proportion becomes 2 : 3 :: $1728 : $2592. (See Arts. 319, 320.)

STATEMENT BY CAUSE AND EFFECT.

$$\left\{\begin{array}{c}\text{1st cause.}\\ \text{48 men}\\ \text{working}\\ \text{24 times}\\ \text{10 hours.}\end{array}\right\} : \left\{\begin{array}{c}\text{2d cause.}\\ \text{60 men}\\ \text{working}\\ \text{36 times}\\ \text{8 hours.}\end{array}\right\} :: \left\{\begin{array}{c}\text{1st effect.}\\ \\ \text{\$1728.}\\ \\ \\ \end{array}\right\} : \left\{\begin{array}{c}\text{2d effect.}\\ \\ \text{\$?.}\\ \\ \\ \end{array}\right\}$$

Rule.—*Write for the third term that number which is of the same kind as the number sought.*

Of the remaining numbers, arrange those of the same kind in ratios, as in Simple Proportion.

Multiply together all the numbers in the means, and divide the result by the product of the numbers in the first extreme.

NOTE.—This rule is sometimes called "*The Double Rule of Three.*"

EXAMPLES FOR PRACTICE.

NOTE.—The following questions should be solved by Analysis, by Simple Proportion, and by Compound Proportion.

2. If 36 men in 24 days of 10 hours each, can grade a street 100 rods long, and 60 feet wide, how many rods of street, 50 ft. wide, can 60 men grade in 27 days of 8 hours each? Ans. 180.

3. If 8 men earn $300 in 25 days, how much will 15 men earn in 24 days? Ans. $540.

4. If 8 horses eat 12 bu. of oats in 6 days, how many horses will eat 60 bu. in 30 days? Ans. 8.

5. If 480 bu. of oats will last 40 horses 24 days, how long will 360 bu. last 48 horses at the same rate?

Ans. 15 days.

6. If 15 men in 9 days of 10 hours each cut 216 acres of grass, how many days of 9 hours each will it require 20 men to cut 300 acres? Ans. $10\frac{5}{12}$.

7. If 50 men in 40 days of 10 hours each, build a wall 30 ft. long, 8 ft. high, and 2 ft. thick, how many men will be required to build a wall 40 ft. long, 6 ft. high, and 2 ft. thick, in 20 days of 10 hours each? Ans. 100.

8. How many boards $12\frac{1}{2}$ ft. long and 8.5 in. broad, are equivalent to 4000 boards $16\frac{2}{3}$ ft. long and 4.25 in. broad?

Ans. $2666\frac{2}{3}$.

9. If I gain $345.60 on $2160 in 2 yr. 8 mo., how much would I gain on $2880 in 3 yr. 6 mo.?

10. If a block of marble 8 ft. long, $4\frac{2}{3}$ ft. wide, and 3 ft. 6 in. high, cost $36, what will be the cost of a block 8 ft. 4 in. long, 4.5 ft. wide, 3 ft. high? Ans. $$34\frac{5}{7}$.

11. If 8 barrels of apples buy 250 bushels of coal, and 100 bushels of coal buy 4 pairs of shoes, how many such pairs of shoes would 48 barrels of apples buy? Ans. 60.

STATEMENT.

100 bu. : 250 bu. : : 4 pr.
8 bbl. : 48 bbl.

EXPLANATION.

Questions of this kind differ somewhat, in their form of being propounded, from such as have been presented, but are capable of solution by Analysis, Simple Proportion, or Compound Proportion. Such forms of questions have sometimes been classed as *conjoined proportions*, and solved by a special rule called the *Chain Rule*. The first ratio given is explained thus:—If 100 bu. buy 4 pairs of shoes, 250 bu. will buy more. The second is explained thus:—If 8 bbl. buy a certain number of pairs of shoes, 48 bbl. will buy more. After stating, solve as usual.

12. If 12 barrels of flour are worth 50 bushels of wheat, and 70 bushels of wheat are worth 100 bushels of corn, how many bushels of corn will buy 100 barrels of flour?

Ans. $595\frac{5}{21}$.

13. If 6 yd. of broadcloth will buy 15 yd. of silk, and 7 yd. of silk will buy 30 yd. of linen, how many yards of linen can be bought for 20 yd. of broadcloth? Ans. $214\frac{2}{7}$.

14. If 50 men in 45.5 days of 9.5 hours each, can do a piece of work, in how many days of 8.5 hours each, can 57 men do the same? Ans. $44\frac{31}{51}$.

15. If 30 men in 27 da. of $9\frac{3}{4}$ hr. each, can dig a ditch 50 rd. long, $2\frac{1}{2}$ ft. wide, and $4\frac{1}{3}$ ft. deep, what length of ditch $2\frac{2}{3}$ ft. wide, and $4\frac{1}{2}$ ft. deep, can 40 men dig in 36 da. of 10 hr. each, the work in the latter case being twice as hard as that in the former? Ans. $46\frac{22}{243}$ rd.

DISTRIBUTIVE PROPORTION.

Art. 330. Distributive Proportion is proportion applied to the distribution of a quantity into parts which have a given ratio. It is also called *Partitive Proportion.*

Art. 331. To resolve a number into parts having a given ratio.

Ex. 1. Resolve 48 into two parts, of which the first shall be to the second as 3 to 5. Ans. 18 and 30.

WRITTEN PROCESS.

$3 + 5 = 8)48(6$

$3 \times 6 = 18$, the first part.

$5 \times 6 = 30$, the second part.

BY PROPORTION.

$3 + 5 : 3 :: 48 : 18$, the less.

$3 + 5 : 5 :: 48 : 30$, the greater.

EXPLANATION.

Since the required parts of 48 are to each other as 3 units and 5 units, 48 must contain $3 + 5 = 8$ of those units. Hence, one of those units is $\frac{1}{8}$ of 48, or 6. One part is, therefore, 3 times 6, or 18, and the other is 5 times 6, or 30.

Rule.—*Divide the given number by the sum of the terms of the ratio, and multiply the quotient by each term.* Or,

As the sum of the terms of the ratio is to one of them, so is the number to be resolved to its corresponding part.

EXAMPLES FOR PRACTICE.

2. Two numbers are to each other as 5 to 7, and their sum is 216; what are the numbers? Ans. 90 and 126.

3. A and B bought a farm for \$15600. A paid \$4 as often as B paid \$9. In dividing it, how much ought each to receive, the land being worth \$60 per acre?

Ans. A 80 A. B 180 A.

4. What parts of 1270 are as $4\frac{1}{4}$ and $6\frac{1}{3}$?

Ans. 510; 760.

5. What parts of 162.5 are as $2\frac{1}{2}$, $3\frac{1}{3}$, $4\frac{1}{4}$, $6\frac{1}{6}$?

Ans. 25; $33\frac{1}{3}$; 42.5; $61\frac{2}{3}$.

6. What parts of $37\frac{1}{2}$ are as $1\frac{1}{2}$, $2\frac{3}{4}$, $3\frac{4}{5}$, $5\frac{7}{12}$?

7. Resolve 63 into two parts, one of which shall be six times the other. Ans. 9 and 54.

Explanation.—If the greater part is 6 times the less, both parts are 7 times the less, and the less is $\frac{1}{7}$ of 63, or 9, and the greater is $6 \times 9 = 54$.

8. Resolve 84 into two parts, one of which shall be three times the other. Ans. 21 and 63.

9. Divide \$870 among A, B, and C, so that A shall have 3 times as much as B, and B 7 times as much as C.

Ans. A \$630, B \$210, C \$30.

10. A and B earned \$84; A earned $8\frac{1}{3}$ times as much as B. How much did each earn?

11. John is $\frac{1}{3}$ as old as his father, and the sum of their ages is 64 years; how old is each?

12. Resolve 19.5 into two parts, one of which shall be $2\frac{1}{4}$ times the other.

13. Resolve 75 into 2 parts, one of which is 15 less than the other. Ans. 30 and 45.

Explanation.—If 75 equals the less plus the less plus 15, 75 is 15 more than twice the less. Hence, 75 — 15, or 60, is twice the less, and $\frac{1}{2}$ of 60, or 30, is the less, and 30 + 15, or 45, is the greater.

14. Resolve 108 into two parts, one of which is 28 less than the other. Ans. 40 and 68.

15. Resolve 162 into two parts, one of which is 24 greater than the other.

16. E and F have 328 A. of land; how many acres has each, if E has 38 A. less than F? Ans. E 145, F 183.

17. Harry and Oscar picked 130 qts. of cherries; how many quarts did each pick, if Harry picked 16 quarts more than Oscar?

18. Resolve 72 into two parts, such that 5 times one equals 4 times the other. Ans. 32 and 40.

EXPLANATION.—If 5 times the first equals 4 times the second, the first equals $\frac{1}{5}$ of 4 times the second, that is, $\frac{4}{5}$ of the second. But the second is $\frac{5}{5}$ of itself. Hence $72 = \frac{4}{5} + \frac{5}{5} = \frac{9}{5}$ of the second. Hence $\frac{1}{5}$ of the second is $\frac{1}{9}$ of 72, or 8, and the second is 5×8, or 40; and the first is 4×8, or 32.

19. Resolve 140 into two parts, such that 3 times the less shall equal twice the greater. Ans. less 56, greater 84.

20. Five times A's age equals 7 times B's, and the sum of their ages is 84 years; what is the age of each?

21. I have a log 114 ft. long, which I wish to cut into two pieces, so that 15 times one piece shall equal 4 times the other; how long must I cut each piece?

22. Resolve 93 into two such parts that $\frac{3}{4}$ of one equals $\frac{4}{5}$ of the other. Ans. 48 and 45.

EXPLANATION.—If $\frac{3}{4}$ of the first equals $\frac{4}{5}$ of the second, $\frac{1}{4}$ of the first equals $\frac{1}{3}$ of $\frac{4}{5}$, or $\frac{4}{15}$ of the second, and the first equals $4 \times \frac{4}{15} = \frac{16}{15}$ of the second. Hence $93 = \frac{16}{15} + \frac{15}{15} = \frac{31}{15}$ of the second, and $\frac{1}{31}$ of 93, or 3, is $\frac{1}{15}$ of the second, and 15×3, or 45, is the second, and $\frac{16}{15}$ of 45, or 48, is the first.

23. Divide $850 between A and B, so that $\frac{2}{3}$ of A's share shall equal $\frac{3}{4}$ of B's. Ans. A $450, B $400.

24. If $\frac{3}{5}$ of A's age equals $\frac{2}{7}$ of B's, and the sum of their ages it 62 years, what is the age of each?

Ans. A 20 and B 42 years.

25. A tree 95 feet high in falling broke into two pieces, so that $\frac{4}{5}$ of the longer piece equals $\frac{8}{9}$ of the shorter; how long was each?

Synopsis of Analysis.

Analysis.
- KINDS.
  - By aliquot parts.
  - General.
    - MEANS.
      - By proportion.
      - By equations.

Synopsis of Ratio.

Ratio.
- KINDS.
  - Arithmetical.
  - Geometrical.
    - KINDS.
      - Simple.
      - Compound.
      - Duplicate, &c.
        - KINDS.
          - Direct.
          - Inverse.
          - Equality.
          - Greater inequality.
          - Less inequality.
            - VARIATION.
              - Direct.
              - Inverse.
              - Proportional.
              - By Addition.
              - By Subtraction.

Synopsis of Proportion.

Proportion.
- KINDS.
  - Simple.
  - Compound.
  - Distributive.
  - Conjoined.
- TERMS.
  - Extremes.
  - Means.
  - Homologous.
  - Identical, of two nos.
  - Of three nos.
  - Of four or more nos.
- VARIATION BY
  - Alternation.
  - Inversion.
  - Composition.
  - Conversion.
  - Equal multiplication, or division of terms of Ratio.
  - Equal multiplication, or division of homologous terms.
  - Multiplication or division of corresponding terms of proportions.

CHAPTER XVI.

PERCENTAGE AND ITS APPLICATIONS.

Art. 332. **Percentage** is the name of that department of Arithmetic which treats of computing by hundredths.

A certain **percentage,** or **per cent.** of a quantity, is so many hundredths of that quantity. Thus, 6 per cent. of a person's income is 6 hundredths of it.

NOTE.—The term *per cent.* is a contraction of the Latin *per centum,* which signifies *by the hundred.*

Art. 333. The essential terms connected with operations in percentage are *rate per cent., base,* and *percentage.*

The **rate per cent.** is the number of hundredths.

The **base** of percentage is that number, of which a certain per cent. is computed.

The **percentage** is the quantity found by computing a certain per cent. of the base.

The non-essential terms, sometimes used, are *amount* and *difference.*

The **amount** is the sum of the base and percentage.

The **difference** is the difference of the base and percentage.

Art. 334. The notation of rate per cent. is by three methods, namely, by common fractions, decimals, and the sign %. Thus, 6 per cent. may be written $\frac{6}{100}$, or .06, or 6%. The sign % is read "*per cent.*" Thus, 5% is read "*Five per cent.*"

The sign is generally used in mere written expressions, and the decimal and fractional methods in calculations.

Art. 335. From the definition of per cent. it follows that

One per cent. of a quantity is $\frac{1}{100}$ of it.

Less than one per cent. of a quantity is less than $\frac{1}{100}$ of it.

One hundred per cent of a quantity is that quantity.

Two hundred per cent. of a quantity is twice that quantity, &c.

NOTE.—The expressions *one-half per cent., one-fourth per cent., &c.,* mean one-half of one per cent., one-fourth of one per cent., &c.

EXERCISES.

Write, in three ways, 1 per cent.; 4 per cent.; 7 per cent.; 10 per cent.; 12 per cent.; 25 per cent.; 50 per cent.; 3 per cent.; 30 per cent.

Write in pure and mixed decimal form $6\frac{1}{4}$ %.

Ans. .0625; .06$\frac{1}{4}$.

Write in like manner $12\frac{1}{2}$ %; $18\frac{3}{4}$ %; $31\frac{1}{4}$ %; $37\frac{1}{2}$ %; $\frac{1}{2}$ %; $\frac{1}{4}$ %; $\frac{3}{4}$ %; $\frac{1}{8}$ %; $7\frac{1}{8}$ %; $\frac{3}{8}$ %; $5\frac{3}{8}$ %; $87\frac{1}{2}$ %; $\frac{5}{8}$ %; $62\frac{5}{8}$ %; $\frac{3}{10}$ %.

Write, in three ways, 125 per cent. Ans. $\frac{125}{100}$; 1.25; 125%.

Write in like manner 100 per cent.; 110 per cent.; 250 per cent.; 375 per cent.; 411 per cent.; 1000 per cent.; 650 per cent.

CALCULATIONS IN PERCENTAGE.

CASE I.

Art. 336. To find any percentage of a number.

Ex. 1. Find 7% of \$125. Ans. \$8.75.

FIRST METHOD.—\$125 ÷ 100 = \$1.25; \$1.25 × 7 = \$8.75.

SECOND METHOD.—\$125 × .07 = \$8.75.

BY PROPORTION.—100 : 7 : : \$125 : \$8.75. Or, 1 : .07 : : \$125 : \$8.75.

Rule.—*Multiply one-hundredth of the number by the number of hundredths.* Or,

Multiply the number by the rate expressed decimally. Or,

Find that part of the number which the rate is of 100.

NOTE.—By the last method it is often best to use lowest terms. Thus, *5% is $\frac{1}{20}$;* 10% is $\frac{1}{10}$; $6\frac{1}{4}$% is $\frac{1}{16}$; 20% is $\frac{1}{5}$; 25% is $\frac{1}{4}$; 50% is $\frac{1}{2}$; &c.

EXAMPLES FOR PRACTICE.

Find	Ans.	Find	Ans.
2. 4 per cent. of 350.	14.	9. $16\frac{2}{3}\%$ of 186 da.	31 da.
3. 5 per cent. of 4500.	225.	10. 80% of $725.	$600.
4. 7 per ct. of 152.5.	10.675.	11. 125% of 125 lb.	156.25.
5. $8\frac{1}{2}$ per ct. of $216.	$18.36.	12. 218% of 75.	
6. 9 per cent. of 355 mi.		13. $\frac{5}{8}\%$ of 1728.	10.368.
7. 25 per cent. of 187 bu.		14. $\frac{1}{2}\%$ of 63.	.315.
8. $18\frac{3}{4}$ per cent. of 432.		15. $\frac{4}{5}\%$ of $\frac{7}{8}$.	$\frac{7}{1000}$.

16. What is $6\frac{2}{3}\%$ of 147 yd.? Ans. 9.8 yd.

17. A man having 1250 bu. of wheat, sold $2\frac{2}{5}\%$ of it; how much did he sell?

18. My salary is $1200; If I pay 18% for board, $7\frac{1}{2}\%$ for clothing, $1\frac{1}{2}\%$ for travelling expenses, $3\frac{2}{3}\%$ for books, and $8\frac{1}{3}\%$ for incidentals, what are my yearly expenses?

Ans. $468.

CASE II.

Art. 337. To find the rate per cent. which one number is of another.

Ex. 1. What per cent. of $20 is $2.50? Ans. $12\frac{1}{2}\%$.

FIRST METHOD.—One per cent. of $20 is $0.20, and $2.50 is as many per cent. of $20, as $0.20 is contained times in $2.50, that is, $12\frac{1}{2}\%$.

SECOND METHOD.—Since $20 is 100% of itself, and $2.50 is $\frac{1}{8}$ of $20, $2.50 is $\frac{1}{8}$ of 100%, or $12\frac{1}{2}\%$ of $20.

BY PROPORTION.—$20 : $2.50 : : 100 : $12\frac{1}{2}$, or $20 : $2.50 : : 1 : .125.

Rule.—*Divide the percentage by 1% of the base.* Or, *Find that part of 100 which the percentage is of the base.*

EXAMPLES FOR PRACTICE.

What per cent.	Ans.	What per cent.	Ans.
2. Of 25 is 15?	60.	6. Of $100 is $.25?	$\frac{1}{4}$.
3. Of 15 is 25?	$166\frac{2}{3}$.	7. Of 2 bu. is 150 bu.?	7500.
4. Of $125 is $6.25?	5.	8. Of $12\frac{1}{2}$mi. is $8\frac{1}{3}$mi.?	$66\frac{2}{3}$.
5. Of 155 A. is 173.6 A.?	112.	9. Of $\frac{3}{4}$ is $\frac{2}{3}$?	$88\frac{8}{9}$.

	What per cent.	Ans.		What per cent.	Ans.
10.	Of 152.5 is 10.675?		14.	Of \$250 is \$30?	
11.	Of 216 is 18.36?	$8\frac{1}{2}$.	15.	Of \$30 is \$250?	$833\frac{1}{3}$.
12.	Of 186 T. is 31 T.?		16.	Of $\frac{4}{5}$ is $\frac{3}{10}$?	$37\frac{1}{2}$.
13.	Of 147 ft. is 9.8 ft.?	$6\frac{2}{3}$.	17.	Of \$500 is \$.40?	$\frac{2}{25}$.

Case III.

Art. 338. To find the base from the rate and percentage.

Ex. 1. Three dollars is 6% of what? Ans. \$50.

First Method.—$\$3 \div .06 = \$50$. Or, $\$3 \div \frac{6}{100} = 0.50 \times 100 = \$50$.

Analysis.—If $\frac{6}{100}$ of a quantity is \$3, then $\frac{1}{100}$ of that quantity is $\frac{1}{6}$ of \$3, or $\$\frac{3}{6}$, and $\frac{100}{100}$, or the whole of the quantity, must be 100 times $\$\frac{3}{6}$, or $\$\frac{300}{6}$, which equals \$50.

Second Method.—$100\% \div 6\% = 16\frac{2}{3} : 16\frac{2}{3} \times \$3 = \$50$.

Analysis.—Since \$3 is 6% of a quantity, that quantity must be as many times \$3 as 100% contains 6%, that is, $16\frac{2}{3}$ times \$3, or \$50.

By Proportion.—6% : 100% : : \$3 : \$50.

Rule.—*Find as many times the percentage as 100 contains the rate.* Or,

Divide the percentage by the rate expressed as hundredths.

Examples for Practice.

2. 23 is 20% of what number? Ans. 115.
3. 40 is 16% of what number?
4. 19.5 is $4\frac{1}{3}$% of what number? Ans. 450.
5. 1530 is $127\frac{1}{2}$% of what number?
6. \$.40 is $\frac{2}{25}$% of how many dollars? Ans. \$500.
7. I spent \$198.75, which was 15% of my salary; what was my salary? Ans. \$1325.
8. \$10.24 is $\frac{4}{5}$% of how much?
9. \$875 is 216% of what?
10. I sold 48 sheep, which was $5\frac{1}{3}$% of my flock; how many had I at first, and how many had I left?

Ans. At first, 900. Left, 852.

CASE IV.

Art. 339. To find the base from the rate and amount, or from the rate and difference.

Ex. 1. What number, increased 25%, equals $50?

Ans. $40.

METHOD INDICATED.—$50 ÷ 1.25 = $40. Or, $50 ÷ $\frac{125}{100}$ = $40.

ANALYSIS.—Since $50 equals $\frac{100}{100} + \frac{25}{100}$, or $\frac{125}{100}$, of the number, that is, 1.25 times the number, $\frac{1}{100}$ of the number is $\frac{1}{125}$ of $50, or $0.40, and $\frac{100}{100}$ of the number is 100 times $0.40, or $40.

Ex. 2. What number, diminished 25%, equals $30?

Ans. $40.

METHOD INDICATED.—$30 ÷ .75 = $40. Or, $30 ÷ $\frac{75}{100}$ = $40.

ANALYSIS.—Since $30 equals $\frac{100}{100} - \frac{25}{100}$, or $\frac{75}{100}$, of the number, $\frac{1}{100}$ of the number is $\frac{1}{75}$ of $30, or $0.40, and $\frac{100}{100}$ of the number is 100 times $0.40, or $40.

Rules.—I. *Divide the amount by the sum of 1 and the rate expressed as hundredths.*

II. *Divide the difference by the difference between 1 and the rate expressed as hundredths.*

EXAMPLES FOR PRACTICE.

3. What number increased by 18% of itself equals 382.32? Ans. 324.
4. What number diminished by 18% of itself equals 265.68? Ans. 324.
5. $3.60 is $33\frac{1}{3}$% less than how much? Ans. $5.40.
6. $5.40 is $33\frac{1}{3}$% more than what? Ans. $4.05.
7. A merchant during the year gained 12% on his capital, and found he was worth $5180; what was his capital?
8. $59.50 is $16\frac{2}{3}$% greater than what? Ans. $51.
9. $2382 is $\frac{3}{4}$% less than what? Ans. $2400.
10. After spending $35\frac{5}{8}$% of my money, I have $206; how much had I at first?
11. What number increased by 500% of itself equals 5?

Miscellaneous Examples for Practice.

1. What is $3\frac{1}{3}\%$ of $\frac{4}{5}$? Ans. $\frac{2}{75}$.

2. $\frac{4}{5}$ is $3\frac{1}{3}\%$ of what number? Ans. 24.

3. What part of a quantity is 10% of it? $12\frac{1}{2}\%$? $14\frac{2}{7}\%$? $16\frac{2}{3}\%$? $18\frac{3}{4}\%$? 25%? 50%? 75%? 100%? 125%? 150%? 225%? 375%? 1000%?

4. What is 40% of 50% of a number?
Ans. 20% of the number.

5. What is 50% of 60% of 80% of \$325? Ans. \$78.

6. $\frac{3}{4}$ is what % of $\frac{2}{3}$? Ans. $112\frac{1}{2}\%$.

7. A has \$2000, B \$3000; how many % greater than A's money is B's? How many % less than B's money is A's?
Ans. 50%; $33\frac{1}{3}\%$.

8. B's money is 50% greater than A's; then A's money is how many % less than B's? Ans. $33\frac{1}{3}\%$.

9. A man had 24 sheep killed by dogs, which was $5\frac{1}{3}\%$ of his flock; how many sheep had he left? Ans. 426.

10. In a battle 81 men were killed, which was 12% of 25% of the army; how many remained in the army?
Ans. 2619.

11. After spending 40% of my money, I had \$24 left; how much had I at first?

12. John has \$$\frac{4}{5}$, and James \$$\frac{5}{6}$; what % of John's money equals James's? What % of James's money equals John's?
Ans. $104\frac{1}{6}\%$; 96%.

13. A's salary is \$1600. He spends for rent 25% of it, for books 5%, for necessaries 30%, and for charities 10%; how much does he save?

14. A man dying, left $33\frac{1}{3}\%$ of his property to his wife; 50% of the remainder to his son; 75% of the remainder to his daughter; 70% of the remainder to the church, and the remainder, \$300, to his servants. What was the whole property, and each share? Ans. \$12000; wife, \$4000; son, \$4000; daughter, \$3000; church, \$700.

15. A man sold two cows for \$84 each; for one he received 25% more, and for the other 25% less than its value. What was the value of each? Ans. \$67.20; \$112.

16. A man dying, left to his son 60% more than to his daughter; to his wife 25% more than to his son. What was each one's share, the estate being worth \$18400?

Ans. Daughter's, \$4000; son's \$6400; wife's, \$8000.

APPLICATIONS OF PERCENTAGE.

Art. 340. The convenience of reckoning by the hundred has caused it to be extensively used in business. It is a favorite form of estimating ratios and computing quantities in nearly every branch of trade, or commercial occupation. Its principal applications are in Partnership, Bankruptcy, Taxes, Insurance, Commission, Profit and Loss, Interest, Discount, Stocks, Exchange, and Equation of Payments.

PARTNERSHIP.

Art. 341. Partnership is the association of two or more persons in business.

Partners are persons associated in business.

A business association is styled a **company, firm,** or **house.**

Capital, or **stock,** is the property employed in business.

Cash is ready money, or money at command.

A **dividend** is that which is divided among partners. It may be cash, or stock, or anything of value earned by the firm.

An **assessment** is a demand for money, or its equivalent, to be paid by each partner for his share of the expenses or losses of the firm.

The **liabilities** of a company are its debts.

Assets are such portions of property as can be appropriated to paying debts.

A **concern** is a single business enterprise or undertaking. This term may be applied to any operation in business, into which care and expense are put with the hope of success.

Gross receipts are the total receipts or income.

Net proceeds are the proceeds remaining after all necessary deductions.

A **contract** is an agreement between parties in business.

An **article of agreement** is a written contract.

Art. 342. Partnerships are usually formed with articles of agreement, specifying the essential purposes and methods of conducting the business. They are usually dissolved by settling the accounts of the firm, and the termination of the contract by the parties. The introduction of new partners, and the retirement of old ones, are occasions for dissolving partnerships and forming new ones.

Dividends and assessments are usually made according to the nature of the articles of agreement, but the following two cases, usually given in arithmetics, embody the principles of equity in the circumstances supposed.

CALCULATIONS IN PARTNERSHIP.

Case I.

Art. 343. To find each partner's share of profit, or liability, when their stock is employed the same time.

Ex. 1. The capital of the firm of A & B was \$10000, of which A furnished \$4000, and B the rest. How should they share a dividend of \$3500?

Ans. A's share, \$1400; B's \$2100.

First Method.	Second Method.
A's stock $\frac{4000}{10000}$, or $\frac{2}{5}$, of cap.	Div'd $= \frac{3500}{10000}$, or $\frac{7}{20}$, or 35
B's " $\frac{6000}{10000}$, or $\frac{3}{5}$, of "	% of cap.
$\frac{2}{5}$ of \$3500 = \$1400, A's.	35 % of \$4000 = \$1400, A.
$\frac{3}{5}$ of \$3500 = \$2100, B's.	35 % of \$6000 = \$2100, B.

Illustration by Proportion.

\$10000, whole stock : \$4000, A's stock : : \$3500, div'd : \$1400, A's share.

\$10000, whole stock : \$6000, B's stock : : \$3500, div'd : \$2100, B's share.

Rule.—*Divide the sum to be shared by the whole capital; the quotient is the share of one unit of money: then multiply this quotient by the capital of each partner; the product is his share.*

Or, *As the whole stock is to any partner's stock, so is the whole sum to be shared to his share of it.*

PROOF.—The sum of the shares should equal the dividend or assessment.

NOTE.—This case is sometimes called *Simple Partnership*, to distinguish it from the kind described in Case II.

EXAMPLES FOR PRACTICE.

2. A and B gained \$2856. A's capital was \$4000, B's \$3000; required the gain of each?

Ans. A \$1632, B \$1224.

3. C and D gain in 1 year \$5400; the expenses of the firm are \$1800. If C's stock was \$4500 and D's \$3500, what was the gain of each? Ans. C \$2025, D \$1575.

4. A pasture rents for \$150.50; E puts in 9 cattle; F, 16; G, 14; H, 27; I, 20; and they pay in proportion to the number of cattle; what does each pay?

5. Divide 480 into 4 parts that will be to each other as 3, 5, 7, and 9.

6. A, B, and C entered into partnership; A put in \$240, B \$560, and C \$1120. If they lost \$384, what was each partner's loss? Ans. A's \$48, B's \$112, C's \$224.

7. A banking company, of 500 shares, made an assessment of \$18000. How much had C to pay, who owned 15 shares? Ans. \$540.

8. Divide \$1240 into 3 parts that shall be to each other as $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{5}$. Ans. \$600, \$400, \$240.

9. A, B, and C start a bank; A put in \$9000, B \$12000, C \$15000. A is allowed \$1500 a year for attending to their business; their rent and other expenses are \$1000 a year, and their gain \$9700. How much of it ought each to receive?

Ans. A \$3300, B \$2400, C \$3000.

10. A's gain was \$1600; B's, \$2400; C's, \$3200; their investment, \$13500. What was the stock of each?

Ans. A's, \$3000; B's, \$4500, C's, \$6000.

11. Divide \$4250 between A and B, in the ratio of $\frac{2}{3}$ to $\frac{3}{4}$.

CASE II.

Art. 344. To find each partner's share of profit or liability, when their stock is employed unequal times.

Ex. 1. The firm of A & B began by each furnishing \$4000. After 6 months, the firm needing \$2000 more, A agreed to let the concern have it, if it should be reckoned as stock. This was done, and at the end of a year they divided \$5400. Find the share of each? Ans. A, \$3000; B, \$2400.

WRITTEN PROCESS.

A's stock, 4000, $\times$ 12 = 48000, product for 12 mo. use.
" " 2000, $\times$ 6 = 12000, " " 6 " "
Total of A's products, = 60000
B's stock, 4000 $\times$ 12, = 48000
Total products, 108000) 5400.00 (.05 for 1 unit of prod.
60000 $\times$ \$.05 = \$3000, A's share.
48000 $\times$ \$.05 = \$2400, B's share.

BY PROPORTION. $\begin{cases} 108000 : 60000 :: \$5400 : \$3000 \\ 108000 : 48000 :: \$5400 : \$2400 \end{cases}$

EXPLANATION.—Of A's stock, \$4000, employed 12 mo., was equivalent to 12 times \$4000, or \$48000, employed 1 mo.; and \$2000, employed 6 mo., was equivalent to 6 times \$2000, or \$12000, employed 1 mo. In all, A's stock was equivalent to \$60000 employed 1 mo. B's stock was equivalent to \$48000 employed 1 mo. The whole stock was equivalent to \$108000 employed 1 mo. Hence \$1, employed 1 mo., earns $\frac{1}{108000}$ of \$5400, or 5 cts.; and \$60000 earn 60000 times 5 cts., or \$3000, A's share; and \$48000 earn 48000 times 5 cts., or \$2400, B's share.

Rule.—*Multiply each partner's stock by its time of employment, and find that part of the profit, or liability, which each partner's product is of the sum of the products.*

NOTE 1.—The times of employment must be expressed in like units.
NOTE 2.—This case is sometimes called *Compound Partnership*.

NOTE 3.—Partnership settlements do not usually proceed on this method, but according to the specifications of the Articles of Agreement and any principles of law or equity that apply to individual cases. This case is given here for exercise in equity in cases proceeding on the basis supposed.

EXAMPLES FOR PRACTICE.

2. Brown & Way divided $2332.50. B.'s stock was $4200 employed 9 months, and W.'s was $4625 employed 12 months. What was the share of each? Ans. B. $945; W. $1387.50.

3. Grant & Wilson dissolved partnership, paying $2490 liabilities. G's stock was $7500 employed 7 mo.; W's was $4800 employed 1 yr. 3 mo.; what had each to pay?

4. A, B, and C hired a pasture for $69.60; A put in 6 cows for 12 wk., B 7 cows for 10 wk., and C 15 cows for 6 wk. How much ought each to pay? Ans. A $21.60, B $21, C $27.

5. A and B enter into partnership with $2000 each; after 3 mo., A draws out $500, and B puts in $500; at the end of 3 mo. more the same is done; their year's gain is $2160. What should each get? Ans. A $742.50, B $1417.50.

6. In the grading of a road, A furnished 23 men, each of whom worked 15 days; B furnished 32 men, each working 20; for the whole work they received $3201.25. What did A and B each receive?

7. C, D, and E dissolved partnership July 1, 1868, after 15 months' business. C at first put in $4500, and Dec. 1, 1868, $1500 more. D at first put in $5500, and decreased it to $4000 Feb. 1, 1869. E at first put in $3000, and increased it to $5000 Apr. 1, 1869. What share had each of the dividend, $42000? Ans. C $16500, D $14100, E $11400.

8. A and B are partners: A's capital is to B's as 4 to 7; after 5 mo., A withdraws $\frac{1}{2}$ of his, and B $\frac{2}{3}$ of his; how much should each receive of their year's gain, $1280?

Ans. A $510, B $770.

9. A, B, and C are partners, whose stocks are as 4, 5, and 8: after 4 mo. A withdraws $\frac{1}{3}$ of his; after 5 mo. B withdraws $\frac{1}{4}$ of his, and after 6 mo. C withdraws $\frac{3}{8}$ of his; their year's gain is $3662. Divide it.

Ans. A $896, B $1230, C $1536.

10. A and B enter into partnership: A's capital is $2000; B's, $3600; how much must A put in at the end of 8 mo. to entitle him to half of the year's gain? Ans. $4800.

11. C and D are partners; C's capital is $8000; D's, $3000; how much must D put in at the end of 4 mo. to entitle him to half the year's gain?

BANKRUPTCY.

Art. 345. **Bankruptcy,** or **insolvency,** is inability to pay debts, through lack of property.

An **assignee** is a person selected to take charge of the assets of a debtor, and apply them to the payment of the creditors.

Art. 346. To find each creditor's share of the assets of a bankrupt.

Ex. 1. The assignee of Brown & King closed their affairs at an expense of $250, and charged $150 for his trouble. Their assets amounted to $8400, and their liabilities to $16000. How much should S. Boyd receive, whose claim was $3000? Ans. $1500.

WRITTEN PROCESS.

$250 + $150 = $400, expenses of settling.
$8400 — $400 = $8000, net proceeds.
Total claims $16000) $8000.00 ($0.50, div'd on $1, or 50%.
Boyd's claim $3000 × .50 = $1500, his share.
BY PROPORTION.—$16000 : $8000 : : $3000 : $1500.

Rule.—*Find that part of the net proceeds which each creditor's claim is of the sum of the claims.*

EXAMPLES FOR PRACTICE.

2. A man has a farm worth $9000, stock worth $3000, and notes worth $2500; his debts amount to $10150; how much can he pay on $1, and how much will Jones receive, whose claim is $2350? Ans. 70 cts. on $1; Jones $1645.

3. Green, Cooke & Co.'s liabilities were $25000, assets $9700; the expenses of settling, $750. How much did they pay on $1, and what did S. Simpson get, whose claim was $4500? Ans. $.40 on $1; Simpson $1800.

4. The assignee of Dodge & Phelps realized $24425 on their assets, charged $675 as expenses of settling. Their liabilities being $16825, how much did they pay on $1, and how much did C. Meyers get, whose claim was $3250?

5. The assignee of X sells his property for $47364, and charges $300 for expenses of settling. X's debts are, to A $12350, to B $11480, to C $16750, to D $18975, and to E $18885. How much can he pay on $1, and how much will each creditor receive?

TAXES.

Art. 347. A **tax** is a sum of money required from individuals by Government, for public purposes. Taxes are either *direct* or *indirect.*

A **direct tax** is a tax assessed directly upon the property or persons of taxable individuals.

An **indirect tax** is a tax on articles of consumption in their transit from one person to another.

Direct taxes are either *poll-taxes* or *property-taxes.*

A **poll-tax, or capitation tax,** is a tax imposed equally on taxable persons, without regard to their property.

A **poll,** in law, is a taxable person.

A **property-tax** is a tax assessed at a given rate on the value of property.

Property is either *real estate* or *personal property.*

Real estate is fixed property, such as lands, houses, &c.

Personal property includes all kinds of property not fixed, such as goods, furniture, vehicles, live-stock, &c.

An **inventory** is a list of articles of property.

An **assessor** is an officer, one of whose duties is to make a list of taxable property, and its taxable value.

An **assessor's schedule** is a list of taxable property and its taxable value.

CALCULATIONS IN TAXES.

DIRECT TAXES.

Art. 348. To assess a direct property-tax.

Ex. 1. The assessor's schedule of a certain town showes 2000 taxables, and the property assessed at \$1600000. If the poll-tax is to be \$1, and the amount to be raised is \$50000, what is the rate of property-tax? Ans. 3%.

WRITTEN PROCESS.	EXPLANATION.
2000 × \$1 = \$2000, poll-tax. \$50000 — \$2000 = \$48000, prop. tax. \$1600000) \$48000.00 (.03, rate.	If \$1600000 assessed value of property is to pay \$48000, then \$1 is to pay $\frac{1}{1600000}$ of \$48000, or \$0.03.

Rule.—*From the sum to be raised subtract the amount to be raised by poll-tax. Divide the remainder by the assessed value of taxable property: the quotient is the tax on one unit of money.*

Multiply the assessed value of a person's property by the tax on one unit, and to the product add his poll-tax; the result is his tax.

EXAMPLES FOR PRACTICE.

2. The taxable property of a city is \$16400000, the number of polls 8250. A tax of \$389625 is to be raised. If a poll-tax of \$2.50 is levied, what is the rate of taxation, and what is A's tax, who is assessed \$3468, and pays for 2 polls?
Ans. Rate, $2\frac{1}{4}$%. A's tax, \$83.03.

3. What is B's tax on property assessed at \$5180.60, at $1\frac{7}{20}$%, and 3 polls at \$1.50 each? Ans. \$74.43.

4. Taxables, 1645; poll-tax, \$1.50; tax to be raised, \$11777.50; property assessed, \$532000. What is the tax on \$1? What is A's tax, whose property is assessed at \$2318, and who pays for 4 polls? Ans. $1\frac{3}{4}$ cts.; A, \$46.565.

5. B's tax; property, \$320; 1 poll? Ans. \$7.10.

6. C's tax; property, \$1200; 3 polls?

7. What is the assessed value of property taxed \$162.50, at $1\frac{5}{8}$%? Ans. \$10000.

8. A corporation pays $2257.68 tax, at the rate of $1\frac{21}{25}$ %; what is its capital?

9. D pays a tax of $1\frac{7}{12}$ % on his capital, and has left $17006.40; what was his capital, and his tax?

Ans. Capital, $17280; Tax, $273.60.

DUTIES, OR INDIRECT TAXES.

Art. 349. **Duties** are taxes imposed upon merchandise in transportation.

Customs are duties on goods imported or exported.

Excise is an inland duty on articles manufactured or sold.

A **tariff** is a list of duties.

Revenue is income derived from duties and other sources.

A **custom-house** is a house, or office, where is performed the government business concerning imports, exports, and duties.

An **invoice**, or a **manifest**, is a written account of the articles of merchandise transported.

Duties are either *specific* or *ad valorem*.

A **specific duty** is a duty on a definite quantity of an article, without reference to its value.

An **ad valorem duty** is a duty imposed at a certain rate per cent. of the cost of an article.

A **collector** of duties, or taxes, is an officer appointed and commissioned to receive duties or taxes.

A **port of entry** is a place designated by law where customs may be collected.

NOTE.—*Contraband Goods* are goods whose importation or exportation is prohibited by law. *Smuggling* is importing or exporting goods contrary to law.

Art. 350. In estimating duties, deductions or allowances are sometimes made for packages and waste.

Breakage is a deduction for loss by breaking.

Leakage is a deduction for loss by leaking.

Tare is an allowance for the weight of the thing containing the goods.

Draft, or **tret**, is an allowance for waste in handling.

Gross weight is the weight of the goods with the thing which contains them.

Net weight is weight remaining after deductions.

Art. 351. To calculate duties.

Ex. 1. What is the duty on 600 drums of figs, averaging 14 lb. each, at 5 cts. per lb., tare being 10%? Ans. $378.

WRITTEN PROCESS.

600 × 14 lb. = 8400 lb., gross wt.
100% — 10% = 90%, net per cent.
8400 lb. × .90 = 7560 lb., net wt.
7560 × $.05 = $378, duty.

EXPLANATION.

The net weight is 90% of the gross weight: 90% of 8400 lb. = 7560 lb. The duty on 7560 lb., at 5 cts. per lb., is 7560 × $.05 = $378.

Ex. 2. What is the duty at 16%, on an invoice of silk goods which cost $1250? Ans. $200.

WRITTEN PROCESS.

$1250 × .16 = $200.

Rules—I. For specific duties. *Deduct allowances, and multiply the duty on a unit of the article by the remainder.*

II. For ad valorem duties. *Deduct allowances, and find the required percentage of the value of the remainder.*

EXAMPLES FOR PRACTICE.

3. What is the duty on 3 hogsheads of sugar, each weighing 1250 lb., at 3 cts. a pound, tare 12%? Ans. $99.

4. What is the duty on 300 bags of coffee, each weighing 180 lbs. gross, at $2\frac{1}{2}$ cts. a lb., tare $2\frac{1}{2}$%? Ans. $1316.25.

5. What is the duty on 72 barrels of sugar, each weighing 373 lb., gross, at $2\frac{1}{2}$ cts. a lb., tare being $14\frac{2}{7}$%?

6. What is the duty on 43 T. iron, at $36 per T.? Ans. $1548.

7. What is the duty on 200 boxes lemons, invoiced at $2.75 a box, at 12%? Ans. $66.

8. What is the duty on 32 cases of silk, each containing 25 pieces, of 40 yards each, invoiced at $1.80 a yard, at 25% ad *valorem*?

9. What is the duty on 50 crates ware, invoiced at $8350, breakage 10%, at 35% ad valorem? Ans. $2630.25.

10. What will be the duty on 1500 lb. tapioca, invoiced at $5\frac{3}{5}$ cts. per pound, at 15% ad valorem?

11. What is the duty on 800 doz. bottles of Port wine, invoiced at $7.50 per doz., breakage 2%, at 24% ad valorem? Ans. $1411.20.

12. If duty at 15% on an importation was $347.10, what was the invoice of the goods? Ans. $2314.

13. If goods invoiced at $3725 pay a duty of $447, what is the rate of duty? Ans. 12%.

14. If I paid $660 duty on liquor, at 24%, what was it invoiced at, and what did it cost in store?

Ans. Inv. $2750; cost, $3410.

15. The duty on 25 crates crockery, allowing 10% breakage, was $1282.50, at 30% ad valorem; what was the invoice price?

16. English cloths, after paying 15% duty, and other charges to the amount of $96.60, cost in store $8876.85; what was the invoice price? Ans. $7635.

17. The cost in store of a quantity of French leather is $13936; duty 15%, damages allowed by the collector 40%, other charges $65.75; what did the leather cost in France? Ans. $12725.

INSURANCE.

Art. 352. Insurance is security against pecuniary loss, pledged by one party to another. It is also called *assurance*.

The **premium** is the sum paid for insurance.

The **policy** is the written contract between the parties.

Any source of pecuniary loss is subject to insurance, such as by fire and the injuries attending it, the dangers of transportation, robbery, one's personal injury by accident, sickness, or death, or that of those in his employ, or of his live-stock, &c.

Fire insurance is security against loss by fire and the damage attending its extinguishment.

Marine insurance is security against loss by navigation.

Accident insurance is security against loss by personal injury caused by accidents.

Health insurance is security against loss by sickness or disability.

Life insurance secures a specified sum of money to specified survivors, or to heirs, on the death of the party insured.

Endowment insurance secures a specified sum to the person insured on attaining a specified age, or to his heirs, if his death occurs before that age.

The **rate** of insurance is usually a percentage of the value insured. In any case it is based on the knowledge and experience of the insuring party concerning the amount of risk involved in that case. Pledging insurance to another is called *taking a risk*, and the party that does it is frequently called *an underwriter*.

CALCULATIONS IN INSURANCE.

CASE I.

Art. 353. To find the premium.

Rule.—*Multiply the value insured by the rate.*

ILLUSTRATION.—If a house is insured for \$2500, at $1\frac{3}{4}\%$, the premium is $1\frac{3}{4}\%$ of \$2500, that is, $\$2500 \times .0175 = \$43.75$.

EXAMPLES FOR PRACTICE.

Find the premium for insuring

1. A house for \$2450, at 1%. Ans. \$24.50.
2. A barn for \$1800, at $1\frac{1}{4}\%$.
3. A mill for \$11640, at $2\frac{3}{8}\%$.
4. A church for \$23460, at $\frac{2}{3}\%$.
5. Insured $\frac{5}{8}$ of a vessel worth \$32000, and $\frac{4}{5}$ of its cargo worth \$40000, the former at $2\frac{3}{8}\%$, and the latter at $1\frac{7}{10}\%$; *what is the* premium? Ans. \$1064.

6. A hotel was insured for $27000 at $2\frac{1}{3}\%$, the policy costing $5; what was the cost of insurance? Ans. $635.

7. A warehouse was insured for $7500 at $\frac{4}{5}\%$, the policy costing $3; what was the cost of insurance?

CASE II.

Art. 354. To find the rate per cent. of premium.

Rule.—*Divide the premium by the value insured.*

ILLUSTRATION.—If $43.75 is paid for a risk of $2500 on a house, the rate per cent. is $43.75 ÷ 2500 = .0175, or $1\frac{3}{4}$ per cent.

EXAMPLES FOR PRACTICE.

What is the rate per cent. when the value insured

1. Is $32000, prem. $400? Ans. $1\frac{1}{4}\%$.	6. Is $1750, prem. $14? Ans. $\frac{4}{5}\%$.
2. Is $350, prem. $5.25?	7. Is $1890, prem. $66.15?
3. Is $750, prem. $5.625? Ans. $\frac{3}{4}\%$.	8. Is $750, prem. $33.75?
4. Is $2500, prem. $15.62½?	9. Is $85000, prem, $510?
5. Is $15000, prem. $300?	10. Is $14000, prem. $52.50?

CASE III.

Art. 355. To insure property and expenses of insurance.

Ex. 1. If a planing-mill, costing $20000, is insured at 10%, what face of policy would cover the cost, the premium and the cost of the policy, $5? Ans. $22227.77+.

PROCESS INDICATED.

$$20000 + \$5 = \$20005.$$
$$100\% - 10\% = 90\% = .90.$$
$$\$20005 \div .90 = \$22227.77+.$$

ILLUSTRATION BY PROPORTION.

$$90\% : 100\% :: \$20005 : \$22227.77+.$$

EXPLANATION.

Since $1 insured cost 10 ct.,+its share of cost of policy, the insured realizes (in case of loss) 90 ct. — its share of cost of policy. If 90 ct., realized, requires $1 face of policy, then $20000 + $5, realized, requires as many dollars face as 90 cts. are contained times in $20005, that is, $22227.77+.

Rule.—*Add the expenses of insurance to the value of the property, and divide the sum by 1, less the rate.*

Examples for Practice.

What sum covers all losses, when property is worth

2. \$3549, rate $1\frac{1}{3}\%$, and expenses \$3? Ans. \$3600.
3. \$1633, rate $\frac{4}{5}\%$, and expenses \$3.80? Ans. \$1650.
4. \$2400, rate 3%, and expenses \$5? Ans. \$2479.38 +.
5. \$1977.50, rate $\frac{7}{8}\%$, and policy \$5?
6. \$863, rate $\frac{4}{5}\%$, and policy \$5?
7. \$3000, rate $1\frac{1}{2}\%$, and policy \$6?

Art. 356. The insured party often covers a large amount of insurance by taking policies from two or more insurers, so that one may not have too great risks. Also, an underwriter often covers his own risk wholly or partially by reinsuring with another.

Examples for Practice.

Ex. 1. An insurance company, having taken a \$10000 risk at 2%, reinsures one-half of it in another company at $1\frac{1}{2}\%$. If no loss occurs, what does the first underwriter gain? Ans. \$125.

Process Indicated.

\$10000 × .02 = \$200, rec'd.
\$5000 × .015 = 75, paid.
Balance, gain, \$125.

Explanation.

The first underwriter receives a premium of 2% of of \$10000, and pays to the second a premium of $1\frac{1}{2}\%$ of \$5000, or \$125.

2. An insurance company, having taken a risk of \$20000 at $1\frac{1}{4}\%$, reinsured \$8000 at $1\frac{3}{8}\%$ with another company, and \$6000 at $1\frac{1}{3}\%$ with another. If no loss occurs, what does the first company gain? Ans. \$60.

3. A company having taken a risk of \$25000 at $2\frac{1}{5}\%$, reinsured \$9000 at $1\frac{5}{6}\%$, \$8000 at $1\frac{3}{4}\%$, and \$5000 at $1\frac{4}{5}\%$. If no loss occurs, what does the company gain?

4. I took a risk of \$36000 at $1\frac{1}{4}\%$; reinsured \$15000 at $1\frac{1}{5}\%$, and \$9000 at $\frac{9}{10}\%$; what rate of insurance do I get on what is left? Ans. $1\frac{23}{40}\%$.

5. The owner of a vessel wishes to have it insured for \$250000. One company takes a risk of \$80000 at $2\frac{1}{4}\%$; another \$30000 at 3%, another \$100000 at $3\frac{3}{4}\%$, and another the remainder at 4%. What is the rate on the whole \$250000? Ans. $3\frac{11}{50}\%$.

6. I took a risk at $2\frac{1}{2}\%$; re-insured $\frac{3}{8}$ of it at 2%, $\frac{1}{4}$ of it at $1\frac{3}{4}\%$, and $\frac{3}{10}$ of it at $2\frac{3}{4}\%$; what rate of insurance do I get on what is left? Ans. $6\frac{1}{2}\%$.

7. I took a risk at $1\frac{3}{4}\%$; reinsured $\frac{3}{4}$ of it at $2\frac{2}{3}\%$; my share of the premium was \$46.81; how large was the risk? Ans. \$18724.

8. I took a risk at $\frac{9}{10}\%$; reinsured \$10000 at $1\frac{4}{5}\%$, and \$5000 at 1%; my share of the premium was \$95; what sum was insured? Ans. \$25000.

COMMISSION AND BROKERAGE.

Art. 357. Commission is an allowance of pay to an agent for transacting the business of another.

An **agent, or factor,** is a person intrusted with the business of another.

A **commission merchant** is a merchant who buys and sells goods for others, and charges a commission.

A **broker** is an agent who buys and sells stocks, notes, money, &c., and charges a commission.

Brokerage is the commission of a broker.

A **consignment** is merchandise sent to an agent for sale.

A **consignor** is a sender, or maker, of a consignment.

A **consignee** is an agent who receives a consignment.

Commission is usually charged as a per cent. of the money involved in the transaction, but is sometimes charged at a definite price per unit of that which is bought, sold, or collected.

CALCULATIONS IN COMMISSION.

Case I.

Art. 358. To find the commission on a given sum.

Rule.—*Multiply the sum by the rate of commission.*

Examples for Practice.

1. A commission merchant made sales during a year amounting to \$17645, on which he charged 3% commission; what was his commission? Ans. \$529.35.

2. A lawyer charged 6% for collecting a note of \$724.25; what was his fee, and what did he pay over?
Ans. \$43.455, and \$680.795.

3. A farmer paid a broker $\frac{4}{5}$% to invest \$12680 in city bonds; what was his brokerage?

4. My banker buys for me \$15000 worth of bonds, at $\frac{1}{2}$% brokerage. How much do I pay him? Ans. \$15075.

5. My banker sells for me \$15000 worth of bonds, at $\frac{1}{2}$% brokerage. How much do I get? Ans. \$14925.

6. A consignee sells flour for \$7648, at $2\frac{3}{4}$% commission. What is his commission, and what does the consignor receive?

7. I sell 1250 bbl. flour, at \$8.50 per bbl., 45 bbl. potatoes, at \$3.60 per bbl., and 175 bu. wheat, at \$1.60 per bu. What is my commission at $2\frac{1}{4}$%?

8. An agent has $2\frac{1}{2}$% commission, and $3\frac{1}{4}$% for guaranteeing payment; when sales are \$7654.80, what does he get?
Ans. \$440.151.

Case II.

Art. 359. To find the commission, when it is included with the purchase money in a given sum.

Ex. 1. An agent received \$5100, and, after retaining 2% commission, invested the rest. Find his commission, and the amount invested. Ans. Investment, \$5000; Com. \$100.

PROCESS INDICATED.	EXPLANATION.
$\$5100 \div 1.02 = \$5000$, investment. $\$5100 - \$5000 = \$100$, commission. ILLUSTRATION BY PROPORTION. $1.02 : 1 :: \$5100 : \$5000$.	At 2%, the commission on \$1 is 2 cents. Therefore \$1.02 includes \$1 investment and its commission. Therefore \$5100 includes the investment and commission of as many dollars as \$1.02 is contained times in \$5100, that is, \$5000 investment, and \$5100 — \$5000 = \$100, commission.

Rule.—*Divide the given sum by 1 increased by the per cent. of commission; the quotient is the part invested.*

From the given sum subtract the part invested; the remainder is the commission.

EXAMPLES FOR PRACTICE.

2. An agent received \$4305 to invest in wheat; after deducting his commission, $2\frac{1}{2}$%, how much was invested?

Ans. \$4200.

3. A land agent received \$9840 to expend in purchasing land at \$64 an acre; after reserving his commission, 5%, how many acres did he purchase, and what was his commission?

4. What tax must be levied, to yield \$173520.711, and pay $1\frac{1}{5}$% for collection? Ans. \$175628.25.

5. Sold stock on commission at 4%; invested the net proceeds in wool, commission 6%; my whole commission was \$500; what was the commission on each?

Ans. Stock, \$212; wool, \$288.

NOTE.—Stock = 100%. Com. on wool = 4%. Net proceeds = 96%. Com. on wool = $\frac{6}{106}$ of 96%, or $5\frac{23}{53}$%. Both com. = 4% + $5\frac{23}{53}$% = $9\frac{23}{53}$% = \$500.

6. Sold 640 acres of land at \$26.25 per acre, and invested the proceeds in sheep at \$4 a head, after deducting my commissions, 5% for selling, and 5% for buying. How many sheep did I buy, and what were my commissions?

Ans. Sheep, 3800; 1st com., \$840; 2d com., \$760.

7. Sold oats for \$225; invested the proceeds in cloth, after reserving my commissions, $3\frac{5}{8}$% for selling, and 4% for buying. What did I invest, and what were my commissions?

PROFIT AND LOSS.

Art. 360. Profit, or **gain,** is the excess of what is received from the sale of an article over what was paid for it.

Loss is the difference by which that which is received from the sale of an article is less than what was paid for it.

Cost is the amount paid for an article.

Price is the amount asked for an article for sale.

Profit and loss are usually estimated, in mercantile transactions, as a per cent. of the sum invested.

CALCULATIONS IN PROFIT AND LOSS.

Case I.

Art. 361. To find the profit, or loss, when the cost and rate are known.

Rule.—*Multiply the cost by the rate.*

Examples for Practice.

1. I bought a horse for \$160, and sold it at a profit of $18\frac{3}{4}\%$; required the gain. Ans. \$30.

2. A merchant bought 23 bbl. of flour at \$9 per bbl., and sold at a loss of 12%; how much did he lose?
Ans. \$24.84.

3. A man commencing business with \$4000, gained the first year $37\frac{1}{2}\%$, which he added to his capital; the second year he gained $18\frac{2}{5}\%$, which he added to his capital; and the third year he lost $24\frac{3}{4}\%$. How much did he make in the three years? Ans. \$900.28.

4. If cost is \$1200, rate of gain $3\frac{5}{8}\%$, what is the gain?

5. If cost is \$1500, rate of loss $6\frac{2}{3}\%$, what is the loss?

6. If a man sells goods that cost \$1800, at a gain of 100% what is his gain?

7. If I sell 75 horses that cost \$125 per head, at a gain of 24%, 80 cows that cost \$28 per head, at a gain of 15%, and 3600 sheep that cost \$5.50 per head, at a loss of $14\frac{1}{2}\%$, do I gain or lose, and how much?

CASE II.

Art. 362. To find the rate per cent. of gain, or loss, when the cost and gain or loss are known.

Rule.—*Divide the gain or loss by 1 per cent. of the cost.*

Or, *Find such a part of 100 as the gain or loss is of the cost.*

EXAMPLES FOR PRACTICE.

1. A lot was bought for $450, and sold so as to gain $90; what was the gain per cent.? Ans. 20%.

2. A horse was bought for $80, and sold so as to lose $5; what was the loss per cent.? Ans. $6\frac{1}{4}$%.

3. I bought a house for $800, and sold it for $1200; what per cent. did I gain?

4. Bought butter at 25 cts., and sold it at 30 cts.; what per cent. did I gain?

5. If $\frac{2}{3}$ of the cost of an article equals $\frac{3}{4}$ of the selling price, what is the loss per cent.? Ans. $11\frac{1}{9}$%.

6. If $\frac{3}{4}$ of the cost of an article equals $\frac{2}{3}$ of the selling price, what is the gain per cent.? Ans. $12\frac{1}{2}$%.

7. Sugar lost 10% by wastage, and is sold for 30% above cost; what is the gain per cent.? Ans. 17%.

8. Cloth lost 5% by shrinkage, and is sold for 20% above cost; what is the gain per cent.? Ans. 14%.

CASE III.

Art. 363. To find that selling price which realizes a certain per cent. of gain or loss.

Ex. 1. Cost $4; what price gains 25%? Ans. $5.

ANALYSIS.—To gain 25%, that which costs $1 must be sold at $1.25, and that which cost $4 must be sold at 4 × $1.25, or $5.

Ex. 2. Cost $4; what price loses 25%? Ans. $3.

ANALYSIS.—To lose 25%, that which costs $1 must be sold at 25 cents less, namely, 75 cts., and that which cost $4 must be sold at 4 times 75 cts., or $3.

Rule.—*Find the specified per cent. of the cost; add it to the cost in case of gain; subtract it from the cost in case of loss.*

EXAMPLES FOR PRACTICE.

What selling price

3. Gains 12%, on \$84 cost?	7. Gains 200%, on \$200 cost?
4. Loses 12%, on \$84 cost?	8. Loses 5%, on 5 cts. cost?
5. Gains $\frac{2}{3}$%, on \$360 cost?	9. Gains $\frac{3}{4}$%, on 50 cts. cost?
6. Loses $\frac{3}{4}$%, on \$240 cost?	10. Loses 6%, on $\frac{3}{4}$ of a ct. cost?

11. Bought a horse for \$135; what must I sell it for to gain $12\frac{4}{5}$%? What to lose 16%?

12. How must I mark cloth that cost \$3.75, to gain 28%?

13. How must I mark cloth which cost \$2.40, so as to fall 10%, and still gain 25%? Ans. $\$3.33\frac{1}{3}$.

14. Find the asking price, when cost is \$1.50, so as to take off 20%, and still make 20%? Ans. \$2.25.

15. What must 1 ask for ribbon that cost 18 cts., so that I can fall 25%, and still make 20%? Ans. $28\frac{4}{5}$ cts.

16. What must I ask for boots which cost \$7.50, that I may fall 16%, and still make 12%?

17. When cost is 12 cts., on what price can I fall 30%, and still make 12%?

CASE IV.

Art. 364. To find the cost from the price and rate.

Ex. 1. Sale at \$1; gain 25%; find cost. Ans. 80 cts.

ANALYSIS.—If $\frac{125}{100}$ of cost is \$1, $\frac{1}{100}$ of cost is $\frac{1}{125}$ of \$1, and $\frac{100}{100}$ or cost, is $\frac{100}{125}$ of \$1, or 80 cts. Hence, $\$1 \div 1.25 =$ cost.

Ex. 2. Sale at \$3; loss 25%; find cost. Ans. \$4.

ANALYSIS.—If $\frac{75}{100}$ of cost is \$3, $\frac{1}{100}$ of cost is $\frac{1}{75}$ of \$3, or \$.04, and $\frac{100}{100}$, or cost, is $100 \times \$.04$, or \$4. Hence, $\$3 \div .75 =$ cost.

Rule.—*Divide the selling price by 1 increased by the gain per cent., or by 1 diminished by the loss per cent.*

EXAMPLES FOR PRACTICE.

3. A person sold a lot for \$335.16, and thereby gained $6\frac{2}{5}$%; what was the cost? Ans. \$315.

4. Sold flour at \$7.01 a barrel, which was a loss of $12\frac{3}{8}$%; required the cost? Ans. \$8.

5. Sold cloth at \$4.20, and thereby gained 20%; how much did it cost?

6. When gold is at a premium of 20%, how much gold can be had for \$720 in greenbacks? Ans. \$600.

7. Sold lace at $37\frac{1}{2}$ cts. per yard, thereby gaining 40% of 50%; what was the cost? Ans. $31\frac{1}{4}$ cts.

8. A person in play lost 25% of his money the first game, 25% of the remainder the second, and 40% of the remainder the third, when he had \$337.50 left; what had he at first? Ans. \$1000.

Case V.

Art. 365. From the rate at a given price, to find the rate at another price.

Ex. 1. If sale at 12 ct. gains 50%, what per cent. would be made at 10 ct.? At 4 ct.?

Process Indicated.	Process Indicated.
\$12 ÷ 1.50 = \$.08, cost.	\$.08 — \$.04 = \$.04 loss at \$.04.
\$.10 — \$.08 = \$.02, new gain.	\$.08) \$.04 (.50
\$.08) \$.02 (.25, new rate.	= 50% loss at 4 ct.

Rule.—*Find the cost; then the actual gain or loss at the proposed price; then the per cent. which this is of cost.*

Examples for Practice.

2. Sale at 18; gain 50%; find rate at 15. Ans. 25% gain.

3. Find rate at 9. Ans. 25% loss.

4. If by selling goods at \$3.75 I gain 25% what would be the rate, if I sold for \$4.10? Ans. $36\frac{2}{3}$% gain.

5. If by selling sheep at \$4.50 I lose 25%, what would I gain by selling at \$7? Ans. $16\frac{2}{3}$%.

6. I sold a house for \$800, thereby losing $16\frac{2}{3}$%; what would have been the rate at \$900? Ans. $7\frac{1}{2}$% gain.

7. If apples at $\frac{1}{2}$ of a cent apiece is 50% gain, what would be the rate at $\frac{4}{5}$ of a cent each? Ans. 140% gain.

8. If lemons at $4\frac{1}{2}$ cts. is 50% loss, what is the rate at 6 cts.?

9. If by selling flour at $9 per barrel I gain 15%, what is the rate at $3\frac{3}{4}$ cts. per pound?

INTEREST.

Art. 366. Interest is pay for the use of money.

The principal is the sum for the use of which interest is paid.

The amount is the sum of the principal and interest.

The rate of interest is a percentage of the principal for a unit of time, as for a year, or a month. Thus, *7 per cent. per annum*, is $\frac{7}{100}$ of the principal as pay for its use one year; *1 per cent. a month* is $\frac{1}{100}$ of the principal as pay for its use one month.

NOTE.—*Per annum* is Latin, signifying *by the year.*

Legal interest is the rate of interest prescribed by law.

Usury is any rate of interest greater than the legal rate.

NOTE.—The laws of States, fixing the rates and conditions of interest, are subject to such changes that a table of rates for any one year might not be correct the next year. All these legal rates and conditions are usually fully stated in bankers' magazines.

Art. 367. Interest, in reference to the mode of computing it, is *simple* or *compound.*

Simple interest is interest reckoned on the principal only.

Compound interest is interest reckoned on the sum of the principal and its unpaid interest.

CALCULATIONS IN SIMPLE INTEREST.

CASE I.

Art. 368. To find the interest and amount, when the principal, rate, and time are given.

Ex. 1. What are the interest and amount of $125.75 for 4 years at 6% per annum? Ans. I. $30.18; A. $155.93.

FIRST METHOD.	SECOND METHOD.
\$125.75, Principal.	1 year's int. = 6% of \$125.75.
.06, Rate.	4 " " = 24% " "
7.545, Int. 1 yr.	\$125.75 × .24 = \$ 30.18.
4, No. of yrs.	Principal = 125.75.
\$30.18, Int. 4 yr.	Amount = \$155.93.
125.75, Prin.	Or, the int. for 4 yr. is 24% of the prin., and the am't is 124% of the principal. \$125.75 × 1.24 = \$155.93.
\$155.93, Amount.	

THIRD METHOD.—At 6%, the interest of \$1 for 1 year is 6 cts., and for 4 years it is 4 times 6 cents, or 24 cents. Therefore the interest of \$125.75 for 4 years is 125.75 times 24 cents, or \$30.18, and the amount is \$125.75 + \$30.18 = \$155.93.

Rules.—I. *Multiply the principal by the rate per cent., expressed decimally, and that product by the time. The result is the interest. Add the interest to the principal. The result is the amount.*

II. *Compute the per cent. for the whole time, and find that per cent. of the principal. The result is the interest. Add the interest to the principal. The result is the amount.*

III. *Find the interest of one unit of money for the whole time, and multiply that by the number of such units in the principal.*

NOTE.—Custom sanctions the reckoning of 30 days a month, and 12 months a year, in computing interest for considerable times. This is not exact, because it considers days as 360ths of a year, whereas they are 365ths of a year. The convenience of the method, however, causes its almost universal adoption in business. Strictly, a month should not be a unit of time in computing interest, as months vary in length, and the time less than a year should be counted in days as 365ths of a year. The laws of different States require different estimates of time.

EXAMPLES FOR PRACTICE.

2. What are the interest and amount of \$325 for 5 years at 7% per annum? Ans. I. \$113.75; A. \$438.75.

3. What are the interest and amount of \$740 for 2 years at $6\frac{1}{2}$% per annum?

4. What are the interest and amount of \$215.75 for 3 years at $7\frac{2}{5}\%$ per annum?

5. What are the interest and amount of \$2000 for $6\frac{1}{2}$ years at $9\frac{3}{5}\%$ per annum? Ans. I. \$1248; A. \$3248.

6. Find the interest of \$327.65 for 9 years at $8\frac{1}{2}\%$.

7. Find the amount of \$712 for $12\frac{1}{2}$ years at 8%.

8. Find the interest of \$913.27 for 9 years at $11\frac{1}{8}\%$.

9. What is the interest and amount of \$4327.50 for 7 years at $14\frac{2}{7}\%$?

10. What are the interest and amount of \$75 for 4 yr. 7 mo. 24 da. at 8% a year? Ans. I. \$27.90; A. \$102.90.

FIRST METHOD BY RULE I.

		\$75, Pr.
30	24	.08, Rate.
12	7.8	\$6, Int. 1 yr.
	4.65 × \$6 = \$27.90, Int.	
		75.
	Am't,	\$102.90

BY RULE III.

Int. \$1 for 1 yr. = \$.08.
4.65 × \$.08 = \$.372, Int. \$1.
75 × \$.372 = \$27.90, " \$75.
75.
Am't, \$102.90

11. What are the interest and amount of \$360 for 2 yr. 5 mo. 20 da. at 6%? Ans. I. \$53.40; A. \$413.40.

12. What are the interest and amount of \$115 for 3 yr. 7 mo. 28 da. at 8%? Ans. I. \$33.68+; A. \$148.68+.

13. What is the interest of \$643.80 for 4 yr. 9 mo. 18 da. at 9%?

14. What is the interest of \$256.50 from July 25, 1869, to Sept. 12, 1872, at 8%? Ans. \$64.239.

15. What is the interest of \$345.27 from May 16, 1868, to Feb. 21, 1873, at $6\frac{4}{5}\%$?

16. What is the amount of \$923.40 from Jan. 10, 1864, to June 18, 1869, at $7\frac{1}{2}\%$?

17. What is the interest of \$200 from Mar. 15, 1867, to Oct. 23, 1870, at 6%? Ans. $\$43.26\frac{2}{3}$.

18. What would be the interest for the same sum for the same time at the same rate, *counting 365 days to the year?*

Ans. $\$43.29\frac{63}{73}$.

19. Find the interest of \$250 for 3 yr. 8 mo. 11 da., at 9% per annum? Ans. \$83.18¾.

SECOND METHOD BY RULE I.

$$
\begin{array}{lr|l}
 & & \$250. \\
1 \text{ mo.} = \frac{1}{12} \text{ yr.} & & \underline{.09} \\
 & 12 & \$22.50 \times 3 = \$67.50, \text{ Int. 3 yr.} \\
 & 30 & \;\;1.875 \times 8 = \;\;15.000, \text{ `` 8 mo.} \\
11 \text{ da.} = \frac{11}{30} \text{ mo.} & & \;\;.0625 \times 11 = \underline{\;\;\;\;\;\;.6875}, \text{ `` 11 da.} \\
 & & \text{Total interest} = \$83.1875.
\end{array}
$$

20. What is the interest of \$576 for 3 yr. 10 mo. 21 da. at 10% per annum?

21. What is the interest of \$876.68 for 7 yr. 7 mo. 7 da. at $8\frac{1}{2}$%? Ans. \$566.54+.

22. What is the amount of \$739.20 for 1 yr. 8 mo. 15 da., at 7%? Ans. \$827.596.

23. What is the interest and amount of \$600 from Dec. 20, 1862, to March 12, 1865, at $7\frac{3}{10}$%?

24. What is the interest and amount of \$960 from Mar. 12, 1870, to Dec. 20, 1872, at 9%?

25. What interest is due to-day on a note of \$750, at 8%, dated July 31, 1871?

SPECIAL METHODS FOR SIX PER CENT. IN U. S. MONEY.

Art. 369. At 6%, the interest of \$1 for 1 year is 6 cents, or *one-half as many cents as months.* Hence the interest of any number of dollars is that number of times one-half as many cents as months in the given time. The interest for 1 Month is $\frac{1}{12}$ of 6 cts., or ½ ct., or 5 mills, or *one-sixth as many mills as days;* and for any number of months it is that number of times one-sixth as many mills as days in the given time.

Rule.—*Call six times the number of years, and half of the number of months,* cents, *and one-sixth of the number of days,* mills, *and multiply the sum of these results by the principal.*

The result is the interest at six per cent. For any other rate per cent., find as many sixths of this interest as the given rate is of six per cent.

Examples for Practice.

1. Find the int. of $400 for 3 yr. 10 mo. 18 da. at 6%.

Written Process.

3×6 cts. $+ \frac{1}{2}$ of 10 cts. $=$ \$0.23
$\frac{1}{6}$ of 18 m. $=$ .003
\$0.233 $\times 400 =$ \$93.20, Ans.

2. Of $500 for 2 yr. 9 mo. 24 da., at 6%. Ans. $84.50.
3. Of $650 for 3 yr. 5 mo. 12 da., at 6%. Ans. $134.55.
4. Of $700 for 4 yr. 8 mo. 6 da., at 6%.
5. Of $1500 for 3 yr. 7 mo. 29 da., at 6%. Ans. $329.75.
6. Of $1000 for 2 yr. 6 mo. 15 da., at 6%.
7. Of $83.25 for 5 yr. 5 mo. 5 da., at 6%.
8. Of $800 for 6 yr. 4 mo. 27 da., at 7%. At 5%.

Written Process.

6×6 cts. $+ \frac{1}{2}$ of 4 cts. $=$ \$0.38
$\frac{1}{6}$ of 27 m. $=$ .0045
Int. at 6% $=$ \$0.3845 $\times 800 =$ \$307.60.

6)\$307.60, Int. at 6%.

\$51.26$\frac{2}{3}$, Int. at 1%.	\$51.26$\frac{2}{3}$, Int. at 1%.
7	5
\$358.86$\frac{2}{3}$, Int. at 7%.	\$256.33$\frac{1}{3}$, Int. at 5%.

Find the amount

9. Of $300 for 4 yr. 8 mo. 18 da., at 8%. Ans. $413.20.
10. Of $480 for 3 yr. 7 mo. 27 da., at 10%. Ans. $655.60.
11. Of $516.84 for 5 yr. 11 mo. 13 da., at 7%.
12. Of $3000 for 1 yr. 2 mo. 12 da., at 9%.
13. Of $1200 for 7 yr. 3 mo. 19 da., at 12%.

14. Of $342 for 3 yr. 3 mo. 3 da., at 5%.
15. Of $287.40 for 5 yr. 5 mo. 20 da., at 4%.
16. Of $500 for 2 yr. 9 mo. 24 da., at 8%.
17. Of $720 for 3 yr. 6 mo. 9 da., at 11%.

Art. 370. The following special method for 6% in U. S. money is popular *for short times.* Since the interest of $1 for 60 days is 1 cent, for any number of days it is as many cents as 60 is contained times in the number of days. Hence, for any number of dollars it is that number of times this quotient. Hence

Rule I.—*Multiply the number of dollars by the number of days, divide the product by 60, and point off two right-hand figures for cents; the rest is dollars.*

ILLUSTRATION.—The interest of $600 for 94 days. at 6%, is thus found:—$600 \times 94 = 56400$; $\div 60 = 940$; $\div 100 =$ $9.40. From this it is plain that, instead of dividing the product by 10 times 6, then by 100, we can divide the principal by 1000, and the product by 6. Hence the following variety of the same method.

Rule II.—*Point off three right-hand figures from the dollars of the principal, multiply the result by the number of days, and divide the product by 6.*

ILLUSTRATION.—The interest of $600 for 94 days, at 6%, is thus found:—$.600 × 94 = $56.400; ÷ 6 = $9.40. At 7% it is $\frac{7}{6}$ of $9.40; at 5% it is $\frac{5}{6}$ of $9.40, &c.

NOTE.—These methods do not apply when days are reckoned 365ths of a year.

EXAMPLES FOR PRACTICE.

Find the amount	Answers.
1. Of $250 for 90 days, at 6%.	$253.75.
2. Of $270 for 125 days, at 6%.	
3. Of $315 for 186 days, at 8%.	$328.02.
4. Of $2000 for 153 days, at 7%.	$2059.50.
5. Of $1800 for 95 days, at 10%.	
6. Of $735 for 42 days, at 9%.	
7. Of $624 from May 10, 1872, to Sept. 25, 1872, at 10%.	646.18\frac{2}{3}$.

8. Of \$1440 from Jan. 6, 1872, to July 3, 1872, at 8%. Ans. \$1497.28.
9. Of \$1260 from Apr. 27, to Aug. 13, same year, at 5%.
10. Of \$72.36 from June 15, to Dec. 15, same year, at $4\frac{1}{2}$%.
11. Of \$526.20 for 78 days, at $5\frac{1}{2}$%.
12. Of \$3000 for 120 days, at 7%.
13. Of \$514.56 for 132 days, at 11%.
14. Of \$5024.40 for 300 days, at $10\frac{1}{2}$%.
15. Of \$6000 for 276 days, at $7\frac{3}{10}$%.

Methods in English Money.

Art. 371. In computing interest in English Money, when the principal is a compound number it can be multiplied by the rate as a whole number, and the product divided by 100, to find the interest for a year, or the principal can be expressed in pounds and the decimal of a pound by Art. 275, and then multiplied by the rate expressed decimally. Thus, the interest of £2 15s. 3d. for 1 yr. at 5%, is either (£2 15s. 3d.) × 5 = (£13 16s. 3d.) ÷ 100 = £0 2s. 9d. 0.6 far.; or, £2.7625 + .05 = £.138125; = £0 2s. 9d. 0.6 far. by Art. 275.

Examples for Practice.

1. What is the interest of £30 10s. 6d. for 5 yr. 3 mo. 15 da., at 6% per annum? Ans. £9. 13s. 10d. .02 far.
2. What is the amount of £40 8s. 9d. for 3 yr. 5 mo. 20 da., at 8% per annum? Ans. £51 13s. 4d. $3\frac{1}{3}$ far.
3. What is the interest and amount of £42 12s. 1.5d. for 2 yr. 7 mo. 25 da., at 7% per annum?
4. What is the interest and amount of £75 13s. 3d. for 4 yr. 9 mo. 18 da., at 10% per annum?

PROMISSORY NOTES.

Art. 372. **A promissory note,** or note of hand, is a written promise by one party to pay a specified sum of money to another.

A **negotiable note** is a note that can be sold or transferred.

A **joint note** is a note signed by two or more persons, who are unitedly held for its payment.

A **joint and several note** is a note signed by two or more persons, who are unitedly and individually held for its payment.

The **maker,** or **drawer,** of a note is the person who signs it.

The **holder,** or **payee,** is the person to whom it is to be paid.

An **indorser** is a person who writes his name upon a note, or other obligation, usually upon its back, thereby becoming responsible for its payment.

The promise of a note is usually either to pay *on demand,* or at a specified time after the date of the note. If no time is mentioned it is payable on demand.

The units of time are usually days, or months, for what are called *short notes,* and months, or years, for what are called *long notes.*

The promise of a note is usually either to pay the bearer, or some person named in the note, or *to his order.* When the note only promises to pay some person named, it is not negotiable. When it promises to pay the bearer, or to the order of some person named, it is negotiable.

The **face** of a note is the sum promised in a note. This sum should be written *in words* in the text of the note.

A note is said to **mature** on the day on which it is due.

When the time of payment is specified in the note, it is customary to allow three more days, called **days of grace.** If the time is stated in months, *calendar months* are reckoned, and the days of grace are reckoned on after the calendar date. Thus, a note of four months, dated Feb. 15, would mature June 15/18. If the time is stated in days, the days of grace are reckoned after the actual number of days has elapsed. Thus, a note of 60 days would have its last day of grace in 63 days.

The payee cannot hold the maker directly for payment, unless the note was given for having received the value of its face, but an innocent holder of such a note can hold the maker for payment.

If not paid at maturity, a notification of indorsers by a legal officer, called a notary, is called *protesting* the note. If the note is not protested, the indorsers are freed from liability to pay it.

PARTIAL PAYMENTS.

Art. 373. **A partial payment** is a payment of part of a note, or other legal obligation. If stated on the back of the instrument in writing by the holder, it is called an *indorsement.*

Different rules are in use for computing interest when partial payments have been made, designed either wholly or partially to avoid the payment of compound interest to the creditor.

RULE OF THE UNITED STATES COURTS.

Art. 374. Settlements are made by the courts of the United States, and of several individual States, by the following

Rule.—*Find the amount to the time when the payment, or the sum of two or more payments, first equals or exceeds the interest. Subtract the payment, or sum of the payments, and proceed with the remainder as before.*

ILLUSTRATION.

$\$3000\frac{00}{100}$. PITTSBURGH, PA., *Feb. 3, 1868.*

For value received, I promise to pay to the order of John Q. Smith three thousand dollars, without defalcation, with interest from date, at 6 per cent. WILLIAM Z. CHAMBERS.

INDORSEMENTS.—Aug. 3, 1868, \$500; Feb. 3, 1869, \$50; Aug. 3, 1869, \$1000.

How much was due at settlement, Aug. 3, 1870?

Ans. $1797.124.

COMPUTATION.

Principal	$3000.
Int. fr. Feb. 3 to Aug. 3, 1868, (6 mo.,) .	90.
Am't due Aug. 3, 1868, ...	$3090.
Payment (greater than the int.) subtracted,	500.
Balance, new principal, ...	$2590.
Int. fr. Aug. 3, 1868, to Feb. 3, 1869, (6 mo.) $77.70, or more than the pay't, $50; hence	
Int. of $2590 fr. Aug. 3, 1868, to Aug. 3, 1869, (1 yr.) $155.40, less than the sum of the payments $50 + $1000 = $1050:	155.40
Am't due Aug. 3, 1869, ...	$2745.40
Sum of last two payments, ...	$1050.00
Balance, new principal, ...	$1695.40
Int. fr. Aug. 3, 1869, to Aug. 3, 1870, (1 yr.)	101.724
Am't due Aug. 3, 1870, ...	$1797.124

Art. 375. Settlements of notes on which partial payments have been made, and of accounts, are often made according to the following rule, frequently called the

COMMERCIAL, OR MERCHANTS' RULE.

Find the amount of the principal at the time of settlement. Then find the amount of each payment from the time it was made until settlement, and subtract the sum of the amounts of the payments from the amount of the principal.

NOTE.—This is also frequently called the "Vermont Rule." Its use is mostly confined to short times.

ILLUSTRATION.

By this rule, what was due at settlement of the example illustrating the United States Rule?

COMPUTATION.

Principal, ...		$3000.
Int. fr. Feb. 3, 1868, to Aug. 3, 1870, ($2\frac{1}{2}$ yr.)		450.
Amount of principal at settlement, . .		$3450.
First payment, ...	$500.	
Int. fr. Aug. 3, '68, to Aug. 3, '70,	60.	
Second payment, . . .	50.	
Int. fr. Feb. 3, '69, to Aug. 3, '70,	4.50	
Third payment, ...	$1000.	
Int. fr. Aug. 3, '69, to Aug. 3, '70,	60.	
Sum of amounts of payments, . . .		$1674.50
Balance due Aug. 3, 1870, ...		$1775.50

PERIODICAL INTEREST.

Art. 376. When interest is considered as due periodically, for instance, annually, if it is paid when due, it is to be subtracted from the amount of the principal at that time. If interest is not paid when due, it is considered a new debt, or principal, drawing simple interest till settlement.

Art. 377. To find what is due at settlement of unpaid principal and periodical interest.

Rule.—*To the principal add the amount, at simple interest, of each unpaid interest from the time it was due until settlement.*

ILLUSTRATION.

1000\frac{00}{100}$. NEW YORK, *Jan. 1, 1868.*

For value received, we jointly promise to pay to James Booth, or order, January 1, 1871, one thousand dollars, with annual interest at seven per cent.

GEORGE STONE,
HENRY SLATER.

Nothing having been paid on this note till maturity, what was then due?

COMPUTATION.

Principal, ...		$1000.00
Annual int. due Jan. 1, 1869, .	$70.	
Int. on $70, unpaid for 2 yr. .	. 9.80	
Annual int. due Jan. 1, 1870, .	$70.	
Int. on $70, unpaid for 1 yr. .	. 4.90	
Annual int. due Jan. 1, 1871, .	$70.	
Amounts of unpaid int. due, ...		224.70
Amount due at settlement, . .		$1224.70

Art. 378. The rule adopted by the Supreme Court of Connecticut has the following points:—

1. Payments made when interest has run a year or more, and those less than the interest, are treated as in the U. S. rule.

2. A payment made within a year from the beginning of any interest draws interest for the rest of that year, if that year does not extend beyond settlement, and its amount must be taken from the amount of the principal for that year. But if that year does extend beyond settlement, the amounts are computed for both principal and payment to settlement. The difference of these amounts is the balance due.

ILLUSTRATION.

By this rule, what was due at settlement of the example illustrating the United States Rule? Ans. $1794.157.

COMPUTATION.

Principal, ...		$3000.000
Int. to Feb. 3, 1869, 1 yr.		180.
Am't to Feb. 3, 1869, ...		$3180.
First payment, ...	$500.	
Int. fr. Aug. 3, '68 to Feb. 3, '69, .	15.	
Second pay't, Feb. 3, '69, . .	50.	
Am't of pay'ts, Feb. 3, '69, ...		$565.
Balance, new principal, ...		$2615.
Int. to Feb. 3, 1870, ...		156.90
Amount, Feb. 3, 1870, ...		$2771.90
Third payment, ...	$1000.	
Int. to Feb. 3, 1870, ...	30.	
Am't of pay'ts, Feb. 3, 1870, ...		$1030.
Balance, new principal, ...		$1741.90
Int. to Aug. 3, 1870, ...		52.257
Am't due Aug. 3, 1870, ...		$1794.157

Art. 379. Partial and periodical payments of either principal or interest give the lender the chance, by reloaning, to get interest on the money paid in by the borrower, and deprive the borrower of the use or interest of his money, unless the lender allows him interest on his payment. The Commercial Rule allows him this till final settlement, as if a partial payment were a loan from the borrower to the lender. The following plan considers every payment a settlement of a loan and of its interest to the date of payment. It is believed to be absolutely just to both borrower and lender, on the plan of simple interest. It is often called the *Present Worth Rule.*

Rule.—*Consider every payment an amount of a certain part of the principal and its interest at the given rate since the date of the commencement of interest.*

Divide the payment by the amount of one unit of money for the time and rate. The quotient is the required part of the principal. (See Art. 382.)

From the whole principal subtract the sum of the parts thus found. The amount of the remainder at settlement is the sum due.

ILLUSTRATION.

Settle by this rule the example given for illustration of the United States rule.

COMPUTATION.

Principal, ...		$3000.000
First payment, $500, ÷ 1.03, the am't of $1 for 6 mo.	$485.436.	
Second payment, $50, ÷ 1.06, the am't of $1 for 1 yr.	$47.169.	
Third payment, $1000, ÷ 1.09, the am't of $1 for 1 yr. 6 mo.	$917.431.	
Sum of parts of principal paid, ...		$1450.036
Balance, unpaid principal, ...		$1549.964
Int. fr. Feb. 3, '68 to Aug. 3, 70, . . .		232.494
Am't due at settlement, Aug. 3, '70, .		$1782.458

EXAMPLES FOR PRACTICE.

Solve every one of the following examples by each of the foregoing rules for partial payments, namely, the United States Rule, the Commercial Rule, the Connecticut Rule, and the Present Worth Rule.

(2.)

6000\frac{00}{100}$. BOSTON, *Jan. 2, 1864.*

For value received, I promise to pay to John Rose, or order, six thousand dollars, without defalcation, with interest from date, at 6% per annum. JOHN C. SMITH.

INDORSEMENTS.—May 24, 1865, $2000; Jan. 6, 1866, $2000; Dec. 10, 1867, $1500.

How much was due at settlement, July 1, 1868? Ans. U. S. $1526.72; Com., 1398.08\frac{1}{3}$; Conn., $1528.278; P. W., $1468.558.

(3.)

3000\frac{00}{100}$. MOBILE, *Dec. 6, 1869.*

For value received, three months after date, I promise to pay James Arnold, or order, three thousand dollars.

GEORGE MAJOR.

INDORSEMENTS.—June 25, 1871, $2000; May 25, 1872, $50; Sept. 25, 1872, $600.

What was due Apr. 25, 1873?

Ans. U. S., $699.116; Com., $669.25; Conn., $699.116; P. W., $688.21+.

NOTE.—Interest begins *at maturity* of the above note, Mar. 9, 1870.

(4.)

4000\frac{00}{100}$. CHICAGO *Sept. 10, 1870.*

Six months after date, I promise to pay to the order of S. Crawford & Son, four thousand dollars. Value received.

T. J. CRAIG.

INDORSEMENTS.—Apr. 10, 1871, $500; July 10, 1871, $1000; Sept. 10, 1871, $800; Dec. 20, 1871, $1200.

What was due May 20, 1872, at 6%?

Ans. U. S. $642.079; Com. 636.53\frac{1}{3}$; Conn. $637.793+; P. W., $639.75.

(5.)

2750\frac{50}{100}$. COLUMBUS, *Nov. 18, 1870.*

Four months after date, we promise to pay Wm. Hays, or order, two thousand seven hundred fifty dollars fifty cents, at the Shoe and Leather Bank, for value received.

MYERS & STEVENSON.

Due Mar. 21, 1871.

CREDITS ON THIS NOTE.—Apr. 10, 1872, $120; June 25, 1872, $800; Dec. 1, 1872, $1000; Mar. 21, 1873, $200.

Balance due July 21, 1873, at 6%?

(6.)

346\frac{36}{100}$. ST. LOUIS, *Mar. 26, 1868.*

Three years after date, I promise to pay James Purdy, or order, three hundred forty-six and $\frac{36}{100}$ *dollars, with interest from date at 6%. Value received.* JOSEPH PRICE.

INDORSEMENTS.—July 20, 1868, $54.75; Apr. 8, 1869, $10; Sept. 26, 1869, $5.50; Jan. 6, 1870, $150.46.

What was due Nov. 2, 1870?

7. Face $2500, date May 10, 1862, rate 8%.

PAYMENTS.—June 15, 1864, $275. Mar. 20, 1865, $600. Nov. 12, 1865, $1000. Dec. 10, 1866, $500.

Balance due May 10, 1867, at 6%?

CASE II.

Art. 380. To find the rate per cent., when the principal, interest, and time are known.

Ex. 1. At what rate per cent. will $300 pay $72 interest in 3 years? Ans. 8%.

ANALYSIS.—At 1% per annum, the interest of $300 is $3 for one year, and 3 times $3, or $9, for 3 years. If $9 is the interest of $300 at 1%, $72 is the interest of $300 for as many per cent. as $9 is contained times in $72, namely, 8%.

Rule.—*Divide the given interest by the interest of the principal at one per cent. for the given time.*

EXAMPLES FOR PRACTICE.

2. At what rate per cent. will \$325 pay \$130 interest in 5 years?

3. At what rate per cent. will \$75 pay \$14.175 interest in 2 yr. 8 mo. 12 da.? Ans. 7%.

4. If the interest of a note of \$100 for 3 yr. 6 mo. 6 da. is \$31.65, what is the rate? Ans. 9%.

5. The interest of a note of \$360 from Feb. 9, 1864, to May 18, 1865, is \$41.31; what is the rate?

6. At what rate per cent. will any principal double itself in 4, 6, 8, 9, 12, 15, and 20 yr.?

Ans. 25%; $16\frac{2}{3}$%; $12\frac{1}{2}$%; $11\frac{1}{9}$%; $8\frac{1}{3}$%; $6\frac{2}{3}$%.

7. At what rate per cent. will \$150 amount to \$205.375 in 3 yr. 8 mo. 9 da.? NOTE.—Am't — Prin. = Int. Ans. 10%.

CASE III.

Art. 381. To find the time, when the principal, interest, and rate per cent. are known.

Ex. 1. In how many years will \$300 gain \$72 interest at 8% per annum? Ans. 3 yr.

ANALYSIS.—At 8% per annum the interest of \$300 for one year is \$24. Therefore \$72 must be the interest for as many years as \$24 is contained times in \$72, namely, 3 years.

Rule.—*Divide the given interest by the interest of the principal for one period of time at the given rate.*

EXAMPLES FOR PRACTICE.

2. In how many years will \$137.50 gain \$28.875 interest at 7% per annum?

3. In what time will \$50 gain \$10.875 interest at 6% per annum? Ans. 3 yr. 7 mo. 15 da.

4. In what time will \$512.60 gain \$25.71 interest at 7% per annum? Ans. 8 mo. 18 da.

5. Principal \$6919.32, interest \$3113.69, rate 6%; what is the time?

6. In what time will any principal double itself at 6%? 7%? 10%? $12\frac{1}{2}$%? 11%?

Ans. $16\frac{2}{3}$ yr.; $14\frac{2}{7}$ yr.; 10 yr.; 8 yr.; &c.

7. In what time will $1883 amount to $2040.23 at 6% per annum? (See NOTE, Art. 380, Ex. 7.)

Ans. 1 yr. 4 mo. 21 da.

8. Principal $200, amount $224, rate 5%; find the time.

CASE IV.

Art. 382. To find the principal when the time, rate, and amount, or interest, are known.

Ex. 1. What principal, at 8%, will gain $72 in 3 years?

Ans. $300.

ANALYSIS.—At 8% the interest of $1 for 3 years is 24 cents. Therefore $72 is the interest of as many dollars as 24 cents is contained times in $72, that is, $300.

Ex. 2. What principal, at 8%, will amount to $372 in 3 yr.?

Ans. $300.

ANALYSIS.—At 8% the amount of $1 for 3 years is $1.24. Therefore $372 is the amount of as many dollars as $1.24 is contained times in $372, that is, $300.

Rule.—*Divide the given interest or amount by the interest or amount of one unit of money for the rate and time.*

NOTE.—This is an instance of finding the base of percentage, (see Art. 368,) and has been applied to partial payments, (see Art. 379,) and to discount, (see Art. 392.)

EXAMPLES FOR PRACTICE.

3. What principal, at 6%, will gain $84.28 in 5 yr. 10 mo.?

Ans. $240.80.

4. What principal, at 6%, will gain $168.30 in 4 yr. 1 mo. 15 da.?

5. What principal, at 6%, will amount to $849.42 in 4 yr. 9 mo. 12 da.?

Ans. $660.

6. What principal will amount to $595.4338 in 5 yr. 4 mo. 24 da., at 7%?

COMPOUND INTEREST.

Art. 383. Interest is said to be *compounded,* when, if not paid when due, it draws interest like a new principal.

Periodical interest may have any period, as a day, a month, three months, six months, a year, &c., and, if compounded, would be compounded at the ends of the periods.

CALCULATIONS IN COMPOUND INTEREST.

CASE I.

Art. 384. To compute compound interest.

Ex. 1. At compound interest at 6% per annum, find the interest of $500 for 2 yr. 4 mo. Ans. $73.036.

COMPUTATION.

Principal,	$500.
Rate,	.06
Interest first year,	$30.00
Principal added,	$500.
Amount first year,	$530.
Rate,	.06
Interest second year,	$31.80
Principal added,	$530.
Amount second year,	$561.80
Rate for 4 months,	.02
Interest for 4 months,	$11.2360
Principal added,	$561.80
Amount for 2 yr. 4 mo.,	$573.036
First principal subtracted,	$500.
Compound interest,	$ 73.036

Rule.—*Find the amount for the first period of time, and make it the principal for the second period, and so on. Subtract the first principal from the last amount; the remainder is the compound interest.*

EXAMPLES FOR PRACTICE.

At compound interest, find the interest and amount

2. Of $3000, at 5%, for 3 yr. 7 mo. 15 da., payable annually. Ans. I. $581.40+; A. $3581.40+.

3. Of $250 for 2 yr. 6 mo. 10 da., at 8%, payable semi-annually. Ans. I. $54.839+; A. $304.839+.

4. Of $200 for 1 yr. 8 mo. 18 da., at 12%, payable quarterly. Ans. I. $45.019+; A. $245.019+.

Art. 385. As long computations of compound interest are laborious, they are generally performed by the aid of tables giving the amount of a unit of money for specified rates and times. As the tables in use are very extensive, as to kinds of periods, number of periods, and rates per cent., a mere fragment of them is here given as a specimen for year periods.

TABLE,

Showing the am't of one unit at comp. int. per annum.

Yr.	3 per cent.	4 per cent.	5 per cent.	6 per cent.	7 per cent.	8 per cent.
1	1.03	1.04	1.05	1.06	1.07	1.08
2	1.060	1.0816	1.1025	1.1236	1.1449	1.1664
3	1.092727	1.124864	1.157625	1.191016	1.225043	1.259712
4	1.125509	1.169859	1.215506	1.262477	1.310796	1.360488
5	1.159274	1.216653	1.276282	1.338226	1.402551	1.469328
6	1.194052	1.265319	1.340096	1.418519	1.500730	1.586874
7	1.229874	1.315932	1.407100	1.503630	1.605781	1.713824
8	1.266770	1.368569	1.477455	1.593848	1.718186	1.850930
9	1.304773	1.423312	1.551328	1.689479	1.838459	1.999004
10	1.343916	1.480244	1.628895	1.790848	1.967151	2.158924
11	1.384234	1.539454	1.710339	1.898299	2.104851	2.331638
12	1.425761	1.601032	1.795856	2.012196	2.252191	2.518170
13	1.468534	1.665074	1.885649	2.132928	2.409845	2.719623
14	1.512590	1.731676	1.979932	2.260904	2.578534	2.937193
15	1.557967	1.800944	2.078928	2.396558	2.759031	3.172169
16	1.604706	1.872981	2.182875	2.540352	2.952163	3.425942
17	1.652848	1.947900	2.292018	2.692773	3.158815	3.700018
18	1.702433	2.025817	2.406619	2.854339	3.379932	3.996019
19	1.753506	2.106849	2.526950	3.025600	3.616527	4.315701
20	1.806111	2.191123	2.653298	3.207135	3.869684	4.660957
21	1.860295	2.278768	2.785963	3.399564	4.140562	5.033833
22	1.916103	2.369919	2.925261	3.603537	4.430401	5.436540
23	1.973587	2.464716	3.071524	3.819750	4.740529	5.871463
24	2.032794	2.563304	3.225100	4.048935	5.072366	6.341180
25	2.093778	2.665836	3.386355	4.291871	5.427432	6.848475

Yr.	3 per cent.	4 per cent.	5 per cent.	6 per cent.	7 per cent.	8 per cent.
26	2.156591	2.772469	3.555672	4.549383	5.807352	7.396353
27	2.221289	2.883368	3.733456	4.822345	6.213867	7.988061
28	2.287927	2.998703	3.920129	5.111686	6.648838	8.627106
29	2.356565	3.118651	4.116135	5.418387	7.114257	9.317274
30	2.427262	3.243397	4.321942	5.743491	7.612255	10.062656
31	2.500080	3.373133	4.538039	6.088100	8.145112	10.867669
32	2.575082	3.508058	4.764941	6.453386	8.715270	11.737083
33	2.652335	3.648381	5.003188	6.840589	9.325339	12.676049
34	2.731905	3.794316	5.253348	7.251025	9.978113	13.690133
35	2.813862	3.946088	5.516015	7.686086	10.676581	14.785344
36	2.898278	4.103932	5.791816	8.147252	11.423942	15.968171
37	2.985226	4.268089	6.081406	8.636087	12.223618	17.245625
38	3.074783	4.438813	6.385477	9.154252	13.079271	18.625275
39	3.167026	4.616365	6.704751	9.703507	13.994820	20.115297
40	3.262037	4.801020	7.039988	10.285717	14.974457	21.724521
41	3.359898	4.993061	7.391988	10.902861	16.022669	23.462483
42	3.460695	5.192783	7.761587	11.557032	17.144256	25.339481
43	3.564516	5.400495	8.149666	12.250454	18.344354	27.366640
44	3.671452	5.616515	8.557150	12.985481	19.628459	29.555971
45	3.781595	5.841175	8.985007	13.764610	21.002451	31.920449
46	3.895043	6.074822	9.434258	14.590487	22.472623	34.474085
47	4.011895	6.317815	9.905971	15.465916	24.045707	37.232012
48	4.132251	6.570528	10.401269	16.393871	25.728906	40.210573
49	4.256219	6.833349	10.921333	17.377504	27.529930	43.427419
50	4.383906	7.106683	11.467399	18.420154	29.457025	46.901612

Art. 386. To compute compound interest by this table.

Rule.—For a whole number of years within the range of the table. *Multiply the amount of one unit by the number of such units in the principal. The result is the amount. From this subtract the principal. The result is the compound interest.*

For a whole number of years greater than the range of the table. *Find by the table the amount for a number of years within the range of the table, and make that a principal for another number of years within the range of the table. So continue till the time is exhausted.*

For a final fraction of a year. *Use the amount of the whole number of years as a principal for the fractional time.*

NOTE.—This table is not applicable to other periods than a year, excepting when a given rate column gives figures which would belong to the required period. Thus, compounding every six months at 6% gives the figures in the 3% column, every number of that column answering to a six months period.

Examples for Practice.

1. Find the compound amount of \$750, for 17 yr. at 6%, payable annually. Ans. \$2019.579+.

2. What is the compound interest of \$2000, for 6 yr. at 5%, payable annually? Ans. \$680.19+.

3. What is the compound amount of \$1250, for 7 yr. at 7%, payable annually?

4. What is the compound interest of \$624, for 6 yr. at 8%, payable annually?

5. If \$5000 is deposited for a child, at birth, at 6% compound interest, payable semi-annually, till it is of age, (21 yr.,) what will it amount to? Ans. \$17303.475.

6. What is the compound interest of \$400, for 2 yr. 9 mo. at 6%, payable annually? Ans. \$69.67.

7. What is the compound interest of \$1600, for 2 yr. 6 mo. 15 da., at 6%, payable semi-annually?
Ans. \$259.475+.

8. What is the compound interest of \$2380, for 3 yr. 5 mo. 20 da., at 8%, payable annually?

9. What is the compound interest of \$3250, for 1 yr. 9 mo. 18 da., at 10%, payable semi-annually?

10. What will \$376 amount to in 3 yr. 8 mo. 15 da., at 6%, payable annually? Ans. \$466.84.

11. What will \$500 amount to in 100 yr., at 6%, payable annually? Ans. \$169651.036+.

12. What is the compound interest of \$1000 for 75 yr., at 5%, payable annually? Ans. \$37832.673+.

13. What is the compound interest of \$450 for 80 yr., at 7%, payable annually?

14. What is the compound interest of \$25 for 71 yr., at 8%, payable annually?

15. What is the compound interest of \$1 for 21 yr., at 12%, payable quarterly?

CASE II.

Art. 387. To find the principal, by the table, when the time, rate, and compound interest, or amount, are known.

Rule.—*Divide the given compound interest by the corresponding tabular amount, less 1.* Or,

Divide the given compound amount by the corresponding tabular amount.

NOTE.—In case of a final fractional period, first find the principal for the beginning of that period by Art. 382.

EXAMPLES FOR PRACTICE.

1. What principal at 3% compound interest will amount to $463.7096 in 5 yr.? Ans. $400.

2. What principal at 6% compound interest will produce $17657.08 in 20 yr.? Ans. $8000.

3. A father dying, left his property to his son, who was 5 yr. 4 mo. 6 da. old. If it was placed on compound interest at 6%, payable semi-annually, till he was of age, (21 yr.,) and amounted to $75677.4216, what was the estate?

4. Compound interest $69.67, rate 6%, payable annually, time 2 yr. 9 mo.; find principal.

CASE III.

Art. 388. To find the rate, by the table, when the principal, time, and compound interest, or amount, are known.

Rule.—*Divide the amount by the principal; the quotient is the amount of one unit of money for the time and rate. If the periods are whole, find the quotient in the table; the rate is at the top. If there is a final fractional period, first find the amounts of the tabular numbers for that fraction, before comparing the quotient with them.*

EXAMPLES FOR PRACTICE.

1. Prin. \$300, am't \$481.7343, time 7 yr.; find rate. Ans. 7%.

2. If \$750 amounts to \$11088.998 in 35 yr., what is the rate? Ans. 8%.

3. At what rate per cent., payable annually, will \$10000 produce \$79336.646, compound interest in 44 yr. 8 mo. 24 da.? Ans. 5%.

CASE IV.

Art. 389. To find the time, by the table, when the principal, rate, and compound interest, or amount, are known.

Rule.—*Divide the amount by the principal. If the periods are whole, find the quotient in the column of the rate, and the time opposite. If there is a final fraction of a period, find the time opposite the tabular number next smaller than the quotient, and add to it that part of a period which the difference between this smaller number and the quotient is of the difference between this smaller and the next larger tabular number.*

EXAMPLES FOR PRACTICE.

1. In what time will \$8000, at 6% compound interest, amount to \$25657.08?

2. In what time will \$400, at 3% compound interest, produce \$63.7096 interest?

3. In what time will \$750, at 8% compound interest, produce \$10338.998 interest?

DISCOUNT.

Art. 390. Discount is a deduction from a price or debt.

A price is discounted when it is reduced to effect a sale.

A debt is discounted when the creditor, for any reason, agrees to take less than the principal.

A note, due in the future, is discounted when it is bought for a less sum than its face.

The **equitable discount,** or **true discount,** of a future obligation, is the interest of its present purchase money, for the time between purchase and maturity.

The equitable, or true, **present worth** of a future obligation is such a sum as, being put at interest for the time before maturity, will amount to the face of the obligation at maturity.

Bank discount is a deduction made by banks on purchasing a future obligation. It is the interest of the face of the obligation deducted from the obligation at the time of purchase.

Art. 391. A **bank** is an institution legally established for the purpose of dealing in money.

A **bank of deposit** is a bank legally empowered to receive and take charge of the money of others.

A **deposit** is money, or its equivalent, intrusted to the keeping of a bank. A **check** is a written order on a bank for money.

A **depositor** is a person who makes a deposit at a bank.

A **bank of discount** is a bank legally empowered to loan money.

A **bank of issue** is a bank legally empowered to pay out its own promissory notes as money. These notes are frequently called *bank-notes*, or *bank-bills.*

In bank discount, the **proceeds,** or **avails,** of an obligation are its face less the discount; that is, the sum received for it.

Art. 392. If days of grace are allowed, (see Art. 372,) discount is made for the time including them. Banks frequently include *the day of discount,* on the plea that the borrower has the use of the money that day. Thus, if a note is drawn for *sixty days after date,* they reckon the *day of date, sixty days,* and *three days of grace,* making 64 days of interest.

Art. 393. To discount a note, or obligation.

Ex. 1. Find, by both equitable and bank discount, the discount and proceeds of the following note, discounted at date, at 6%.

100\frac{00}{100}$. PHILADELPHIA, *June 28, 1871.*

Four months after date, we promise to pay John Ray, or order, one hundred dollars, without defalcation, value received.

BATES & BROWN.

Ans. Due Oct. 28/31: equitable discount $2.009, proceeds $97.991; bank discount $2.05, proceeds $97.95.

EQUITABLE PROCESS. (See Art. 382.)	BANK PROCESS.
$1.0205, am't $1 for 4 mo. 3 da.	Int. of $100 for 4 mo. 3 da., $2.05 Dis.
$100÷1.0205=$97.991, P.W.	$100 — $2.05 = $97.95, Pro.
$100—$97.991=$2.009, Dis.	

Rules.—For equitable discount. *Divide the face of the obligation by the amount of one of its units for the given time and rate; the quotient is the present worth. Subtract the present worth from the face; the remainder is the discount.*

For bank discount. *Find the interest on the face till maturity; this is the discount. Subtract the discount from the face; the remainder is the proceeds.*

EXAMPLES FOR PRACTICE.

2. What is the present worth of $174.46, due 2 yr. 9 mo. hence, at 8%?

3. What is the true discount of $203.8155, due 2 yr. 6 mo. 9 da. hence, at 6%? Ans. $26.81+.

4. What are the proceeds and discount of $7840, discounted at a bank for 4 mo, 15 da., at 6%?

Ans. Pro. $7659.68, Dis. $180.32.

5. What is the difference between the true and bank discount of $720 for 1 yr. 6 mo. 20 da., at 6%, not reckoning days of grace? Ans. $5.73+.

6. A merchant bought goods to the amount of $24800; $8120 of which was on a credit of 3 mo., $8320 on a credit of 8 mo., and the remainder on a credit of 9 mo. What was the cash value of the goods, money being worth 6%?

Ans. $24000.

Find the day of maturity, their time of discount, and proceeds of the following notes:—

7. $\$1650\frac{00}{100}$. PITTSBURGH, *Apr. 3, 1873.*

Sixty days after date, I promise to pay A. C. Bates, or order, one thousand six hundred fifty dollars, value received.

HARVEY MCCUNE.

Discounted May 5, at 8%.

Ans. June 2/5; time of discount 31 da.; Pro. $\$1638.63\frac{1}{3}$.

8. $\$350\frac{60}{100}$. BALTIMORE, *July 9. 1872.*

Four months after date, I promise to pay Geo. Phelps, or order, three hundred fifty $\frac{60}{100}$ dollars, with interest from date, at 6%, value received. S. S. SANSOM.

Discounted Sept. 10, 1873, at 8%.

Ans. Nov. 9/12; 63 da.; Pro. $352.778+.

NOTE.—The base of discount in this note is the amount of $350.60 for 4 mo. 3 da. at 6%.

Art. 394. To make a note, whose proceeds shall be a given sum.

Ex. 1. What must be the face of a note for 90 days, to draw from bank $300, discount at 6%? Ans. $304.723+.

PROCESS INDICATED.

$0.0155, int. $1, 93 da.
$1 —.0155 = 0.9845, proc. $1.
.9845) $300 ($304.723+.

EXPLANATION.

If $.9845 proceeds require $1 of face, $300 proceeds require as many dollars face as .9845 is contained times in $300, that is, $304.723+.

Rule.—*Divide the given proceeds by the proceeds of one unit of money on the given conditions. The quotient is the face.*

EXAMPLES FOR PRACTICE.

What must be the face of a note payable in

2. Sixty da., rate 6%, to draw \$300? Ans. \$303.183+.
3. Ninety da., rate 10%, to draw \$500?
4. Three mo., rate 5%, to draw \$275? Ans. \$278.598+.
5. Four mo., rate 8%, to draw \$1000?
6. I wish to obtain from a bank \$2000 for 30 days, at 10%; for what sum must I give my note? Ans. \$2018.50+.

Art. 395. Since the proceeds are the principal loaned, and the discount is the interest received for the time till maturity, the rate of interest is determined by Art. 396. Thus, by this, the actual rate in Ex. 1, Art. 394, is 6.094+%.

Art. 396. To find the rate of discount which gives a required rate of interest.

Rule.—*Find the interest and amount for the time and rate of interest on one unit of proceeds. The amount is the face, and the interest is the discount, of a note for the given time. Then divide this discount by the interest of the face at 1% for the time. (See Art. 380.)*

ILLUSTRATION.—To discount for 63 days so as to get 10% for my money, \$1 proceeds at 10% for 63 days amounts to \$1.0175. Hence, if \$1.0175 is the face of a note, and .0175 the discount, the rate of discount is .0175 ÷ (1% of 1.0175 for 63 days =) .00178 = 9.83+%.

EXAMPLES FOR PRACTICE.

What rates of discount

1. On 30-day notes yield 8%, 9%, 10%, 11%, 12% interest?
Ans. $7\frac{1423}{1511}$%; $8\frac{3736}{4033}$%; $9\frac{1101}{1211}$%; $10\frac{10790}{12121}$%; $11\frac{293}{337}$%.
2. On 60-day notes yield 6%, 7%, 8%, 9%, 10%, 11%, 12%, 15% interest? Ans. $5\frac{1895}{2021}$%; $6\frac{26766}{44539}$%; $7\frac{451}{507}$%;
3. On 90-day notes yield 6%, 8%, 10%, 12%, 15%, 18%, 20% interest? Ans. To last $19\frac{3757}{97697}$%.
4. On notes running 1 yr., not reckoning days of grace, %, 8%, 10%, 12%, 20% interest?
Ans. $5\frac{35}{53}$%; $7\frac{11}{27}$%; $8\frac{28}{109}$%; $10\frac{5}{7}$%; $16\frac{2}{3}$%.

STOCKS.

Art. 397. **Stock** is the property employed in business. It may include any kind of value possessed and used for business. Stock is also a name applied to property in a public debt, as the bonds issued by governments.

A **corporation** is a company authorized by a charter, or law, to act and be considered as a single individual.

A **charter** is a legal instrument incorporating certain persons, and defining the powers and duties of the corporation.

The stock of a business corporation is usually represented by a certain number of equal parts, called *shares*.

A **share** is one of the equal parts into which a stock is divided.

Stockholders are the owners of the shares of stock.

Par value is nominal value, or the value claimed for themselves on the face of money, notes, stocks, bonds, drafts, &c.

A stock is **at par** when it sells for its nominal value; **above par,** or **at a premium**, when it sells for more; and **below par,** or **at a discount,** when it sells for less.

Premium and discount are reckoned as a percentage of the par value.

An **installment** is a payment of part of the principal of a debt; for example, a payment by stockholders of part of the price of their stock.

Government and other bonds bear a fixed rate of interest specified upon their face. Such bonds are, for brevity, often named from such features. For example, the bonds of the United States bearing 5% interest are called "U. S. 5 per cents." or "U. S. 5's."

Coupon bonds are bonds to which certificates of periodical interest are attached, which certificates are intended to be cut off from the bond and presented for payment when the interest promised in them is due.

A **coupon** is a certificate of interest attached to a bond.

CALCULATIONS IN STOCKS.

Case I.

Art. 398. To find the cost of a given number of shares.

Ex. 1. What cost 25 shares of bridge stock, par value \$100, at 20% premium? 10% discount?

Ans. \$3000; \$2250.

PROCESSES INDICATED.

$1.20 × 100 × 25 = $3000.	.90 × $100 × 25 = $2250.
Or { 25 × 100 = $2500, par.	Or { 25 × $100 = $2500, par.
$2500×20 = 500, prem.	$2500 × .10 = 250, dis.
2500+500=$3000, cost.	2500—250=$2250, cost.

EXPLANATION.—If \$1 is, at 20% premium, worth \$1.20, then \$100 are worth 100 times \$1.20, and 25 shares at that price are worth 25 times as much, or \$3000. Again, if \$1 is, at 10% discount, worth \$.90, then \$100 are worth 100 times \$.90, and 25 shares at that price are worth 25 times as much, or \$2250.

Rule.—*Find the premium or discount on the par value. Add the premium to the par value; subtract the discount from the par value.*

EXAMPLES FOR PRACTICE.

2. What cost 115 shares of bank stock, par value \$100, at $12\frac{1}{2}$% premium? 10% discount?

3. What cost 1000 shares of Pacific Mail stock, par \$50, at $44\frac{3}{4}$% discount?

4. What cost 56 shares railroad stock, par \$100, at $8\frac{1}{4}$% premium, brokerage $\frac{1}{2}$%? Ans. \$6090.

5. What cost 60 shares of telegraph stock, par \$50, at 12% discount, brokerage $\frac{3}{5}$%? Ans. \$2658.

6. Bought 284 shares of oil stock, par \$50, at $\frac{3}{4}$% discount, and sold at $\frac{7}{8}$% premium; how much did I gain?

CASE II.

Art. 399. To find the number of shares that can be bought for a given sum.

Ex. 1. How many shares of bank stock, par $50, can be bought for $1200, at 20% premium? Ans. 20.

ANALYSIS.—At 20% premium, $1 par is worth $1.20, and $50 par are worth 50 times $1.20, or $60. If $60 buy one share, $1200 buys as many shares as $60 is contained times in $1200, that is, 20 shares.

Rule.—*Find the price of one share, and divide the given sum by it.*

EXAMPLES FOR PRACTICE.

2. How many $50 bonds, at $1\frac{1}{2}$% premium, can be bought for $9541? Ans. 188.

3. How many $1000 bonds, at $\frac{3}{4}$% discount, can be bought for $35730? Ans. 36.

4. When gold is $18\frac{5}{8}$% premium over paper money, what is the value of a paper dollar?

5. My agent sold 3000 head of sheep at $7 per head, commission 5%, and invested the proceeds in stock of the L. S. R. R. at $5\frac{3}{10}$% discount, brokerage $\frac{3}{10}$%; how many shares, of $100 each, will I receive? Ans. 210.

CASE III.

Art. 400. To find that price of stock, on which its dividends are a given rate of interest.

Ex. 1. What price of a $1000 5% coupon bond, pays the purchaser 8%, coupons semi-annual? Ans. $625.

ANALYSIS.—If a $25 coupon is 4% of a price, 1% of it is $\frac{1}{4}$ of $25, or $$6\frac{1}{4}$, and 100% of it is 100 times $$6\frac{1}{4}$, or $625.

Rule.—*Find that per cent. of the par value which the fixed interest is of the required interest.*

EXAMPLES FOR PRACTICE.

2. What must I pay for a $2000 mortgage, bearing 6% interest, that I may realize 10% interest? Ans. $1200.

3. What must be paid for a $7\frac{3}{10}$% school bond of $500, to yield the purchaser 10% interest? Ans. $365.

4. What must be paid for a $1000 bond bearing 6% in gold, to yield the buyer 8% in currency, when gold is 125%? Ans. $937.50.

5. How must I invest in 8% bonds, to make 10% interest? 9%? 6%? 5%? Ans. 20% dis., $11\frac{1}{9}$% dis., $33\frac{1}{3}$% prem., 60% prem.

6. What must be paid for 6% bonds to yield 9%? 9% bonds to yield 6%?

CASE IV.

Art. 401. To find the investment that yields a given income.

Ex. 1. What sum invested in 6% securities, at 80% of par, will yield $600? Ans. $8000.

ANALYSIS.—If $1 par pays $.06, to pay $600 will require as many dollars par as $.06 is contained times in $600, that is, $10000. At 80%, $10000 par are worth 80% of $10000, or $8000.

Rule.—*Divide the given income by the per cent which the stock pays, the quotient is the par value of the required stock.*

Multiply the par value by the market price per cent.

EXAMPLES FOR PRACTICE.

2. What sum invested in 5% stocks at 95%, will yield $1000? Ans. $19000.

3. What sum invested in $7\frac{3}{10}$% government bonds at 114%, will yield $1095?

4. What sum invested in 5–20's bearing 6% in gold at $116\frac{5}{8}$, will yield $1407 in currency, gold being $117\frac{1}{4}$%? Ans. $23275.

5. Gold being $121\frac{5}{8}$%, which is the better investment, 5–20's at 117%, or bonds bearing $7\frac{3}{10}$% interest in currency at 98%, and how much? Ans. Latter, $1\frac{1717}{7644}$%.

EXCHANGE.

Art. 402. Exchange is a method of making payment in a distant place by means of drafts.

A **draft** is a written order, directed by one party to another, to pay money to a third party. It is also called a *bill of exchange*.

The **drawer** is the maker, or signer, of the order.

The **drawee** is the party to whom the order is directed.

The **payee** is the party to whom payment is ordered.

Bills of exchange can be bought and sold like negotiable notes, and remitted to and accepted by a creditor as money.

The **buyer, or remitter,** is the purchaser of the bill. He may have it drawn to his own order, or to the order of another, or to bearer. If drawn payable to bearer, it is liable to be paid when presented by a party who has improperly obtained possession of it. If drawn payable to the order of the remitter, he may make it payable to bearer by merely indorsing it with his name; but he may make it payable to the distant creditor by indorsing it with an order to pay that creditor. If the remitter wishes his creditor to have the power to negotiate the bill, he indorses it with an order to pay to the order of the creditor. The creditor, on receiving it, can indorse it in either of the ways just described, with like effects.

Art. 403. In reference to the places of transaction, bills of exchange are *inland* or *foreign*.

An **inland bill of exchange** is one of which the drawer and drawee reside in the same country. This is sometimes called *home* or *domestic exchange.*

A **foreign bill of exchange** is one of which the drawer and drawee reside in different countries.

Art. 404. In reference to their time of payment, bills of exchange are *sight-bills* or *time-bills.*

A **sight-bill** is a bill payable on presentation.

A **time-bill** is a bill payable a certain time after presentation.

NOTE.—The drawers and drawees of a large part of the bills of exchange are now banks. Distant debtors and creditors purchase, and, after remitting, present for payment these bills at banks. In these cases they are mostly sight-bills.

Art. 405. If a bill of exchange is paid by the drawee, it is said to be *honored;* if not, it is said to be *dishonored*

A time-bill is *accepted* when the drawee agrees to pay it at maturity. He usually does this by his signature to the word "Accepted," written across the face of the bill.

On time-bills three days of grace are usually allowed in this country and Great Britain.

A draft, or bill of exchahge, can be protested like a note, if not accepted or honored. (See Art. 372.)

Art. 406. Exchange is *at par, above par,* or *below par,* according as the price of a bill is *at, above,* or *below* its face.

The **rate** of exchange is a percentage of the face of the draft, reckoned as premium or discount.

The **course of exchange** is 1, increased by the rate of premium, or diminished by the rate of discount.

NOTE.—That region which buys of another more than it sells to it, pays a premium for drafts upon it, because the indebted place has not, by its business, created credits enough in the other place to balance its debts, and must send money there at some cost.

That region which sells to another more than it buys from it, gets drafts upon it at a discount, because the former has more drafts upon the latter than it needs to balance its own debts, and the excess must be collected at some cost.

Art. 407. In the practice of foreign exchange, a **set of exchange** consists of three copies of the same bill, which are sent by different conveyances, to avoid inconvenience by miscarriage. When one of this set has been paid, the others are void.

One person is said to *draw on another* when he is the maker of a draft addressed to the other as drawee.

Art. 408. A person may pay a distant creditor by a draft in two ways, namely :—

1. He may remit a draft to him.
2. He may honor a draft drawn on him by the creditor.

CALCULATIONS IN EXCHANGE.

CASE I.

Art. 409. To compute inland exchange.

Ex. 1. Find the cost of the following draft at $\frac{1}{8}\%$ prem.

250\frac{00}{100}$. CHICAGO, *May 17, 1871.*

Please to pay, at sight, to the order of Walter Cook, two hundred and fifty dollars, and charge the same to the DROVERS' BANK OF CHICAGO.

JOHN A. BAKER, *Cashier.* JAMES S. GWINN, *Pres't.*

To the PARK NATIONAL BANK, *Pittsburgh, Pa.*

PROCESS INDICATED.	EXPLANATION.
$\frac{1}{8}\%$ = .00125: 1.00125, course of exch. $250 × 1.00125 = $250.31¼, cost.	If $1 face costs $1.00125, $250 face cost 250 times $1.00125, or $250.3125.

2. Find the cost of the foregoing draft, drawn for "ten days after sight," discounting at 9% per annum.

Ans. $249.50.

PROCESS INDICATED.	EXPLANATION.
Int. $1 13 days = $.00325. $1 — $.00325 = $.99675, proc'ds of $1. .00125, rate of exch. $.998, cost of $1 face. 250 × $.998 = $249.50, Ans.	At bank discount, $1, due in 13 days, is now worth $.99675, and in the draft it is worth $.00125 more, that is, $.998. Hence, $250, on the face, is worth 250 $.998, that is, $249.50.

Rule.—To find the cost of sight drafts. *Multiply the face by the course of exchange.*

Of drafts payable after sight. *Find the proceeds of $1 at bank discount for the time: add the rate of exchange, if at a premium, or subtract it, if at a discount, and multiply the face of the draft by this result.*

To find the face. *Divide the cost by the cost of a unit of the face.*

EXAMPLES FOR PRACTICE.

What costs a sight draft

3. For \$600, at $1\frac{1}{8}\%$ discount? Ans. \$593.25.
4. For \$600, at $\frac{7}{8}\%$ premium? Ans. \$605.25.
5. For \$825, at $\frac{3}{5}\%$ discount?
6. For \$724, at $\frac{3}{4}\%$ premium?
7. For \$587.50, at $\frac{3}{8}\%$ discount? $\frac{5}{8}\%$ premium?

What costs a draft payable after sight

8. 60 da. \$450, dis. 6%, exch. $\frac{3}{4}\%$ pr.? Ans. \$448.65.
9. 30 da. \$600, dis. 9%, exch. $1\frac{1}{4}\%$ dis.? Ans. \$587.55.
10. 90 da. \$760, dis. 8%, exch. $\frac{1}{2}\%$ premium?
11. 10 da. \$2350, dis. 5%, exch. $1\frac{3}{8}\%$ discount?

12. What will be the cost of a draft of \$6400, for 90 days, at 12%, exchange being $2\frac{1}{2}\%$ premium?

13. What will be the cost of a draft of \$550, for 20 days, at $7\frac{1}{5}\%$, exchange being $1\frac{5}{8}\%$ discount?

14. A merchant in Pittsburgh wishes to pay in New York \$4650, and exchange is $1\frac{1}{4}\%$ premium; what will be the cost of the draft? Ans. $\$4708.12\frac{1}{2}$.

15. Find the face of a draft for \$1644.30, exchange being $1\frac{1}{4}\%$ premium. Ans. \$1624.

16. Find the face of a draft for \$631.80, exchange being $2\frac{1}{2}\%$ discount. Ans. \$648.

CASE II.

Art. 410. To compute foreign exchange.

Foreign exchange implies a comparison of the value of the currencies of the countries between which it takes place.

The **par of exchange** between two countries is the value of the currency of one expressed in that of the other.

The **intrinsic value** of a coin is the value of the metal in it.

The **commercial value** of a coin is its value in trade.

The **legal value** of a coin is its value as established by law.

NOTE.—The intrinsic value of any two coinages is found by comparing the amounts of the precious metal in perfect undepreciated specimens of each, reckoning the alloy, usually, of no value. Thus, if an English sovereign of a certain coinage weighs 123.3 grains, of which $916\frac{2}{3}$ thousandths are pure gold, and an American half-eagle weighs 129 grains, of which 900 thousandths are pure gold, then the intrinsic value of that sovereign is found by this proposition:—

$$129 \times .900 : 123.3 \times .916\tfrac{2}{3} :: \$5 : \$4.8675.$$

As the issues of coins of the same name vary in intrinsic value, if governments vary their fineness, the par of exchange will vary. Previous to 1834 a sovereign (£1) was worth $\$4.44\frac{4}{9}$. This value is called the *old par of exchange with England.* Modern coinages of sovereigns have been near \$4.84. The U. S. mint fixes their average value at \$4.84. The sovereigns of Victoria's reign are worth $\$4.86\frac{3}{4}$. Exchange with England is still often computed on $\$4.44\frac{4}{9}$ as the *nominal par*, a premium being reckoned to bring the pound to its present value, so that, when sterling money is really at par, it is quoted at a premium of about $9\frac{2}{5}\%$, because $109\frac{2}{5}\%$ of $\$4.44\frac{4}{9} = \$4.86+$.

Ex. 1. What costs the following sight-bill of £500 on London, at $9\frac{3}{4}\%$ premium? **Ans. \$2438.88+.**

£500 Exchange. NEW YORK, *June 20, 1871.*

At sight of this first of exchange, (second and third of the same date and tenor unpaid,) pay to the order of Adam Mann five hundred pounds. THOMAS ELMORE & CO.

To H. WILKINS & CO., *Bankers, London.*

PROCESS INDICATED.	EXPLANATION.
$\$4.444444 \times 1.0975 = \$4.8777+$. $\$4.877777 \times 500 = \$2438.888+$.	If £1 costs 1.0975 times $\$4.44\frac{4}{9}$, or \$4.87+, £500 cost 500 times \$4.87+, or \$2438.888+.

2. For what face should a New York sight-bill on London be drawn for \$1000 paid, premium being 10%?

Ans. £204 10s. 10d. 3 far.

ANALYSIS.—If £1 face costs 1.10 times $\$4.44\frac{4}{9} = \$4.888+$, then \$1000 buys as many pounds of face as \$4.888+ is contained times in \$1000, that is, £204.545454+ = £204 10s. 10d. 3 far.

Rule.—To find the cost of sight-bills. *Multiply the face by the value of a unit of itself in the currency in which it is bought, and by the course of exchange.*

To find the cost of time-bills. *After reducing the face to the currency in which it is bought, proceed as with inland bills.*

To find the face. *Divide the cost by the cost of a unit of face.*

NOTE.—Extensive tables of the coins of all commercial nations, and of their values in U. S. money, are prepared for the use of business with those nations. As it is not the purpose of an arithmetical text-book to be an encyclopedia of commerce, only a few such statistics are needed for the examples for practice under the rules of exchange. A few are given in the following table, and for the rest the learner is referred to publications devoted to this subject.

U. S. CUSTOM-HOUSE VALUE OF SOME FOREIGN COINS.

Pound Ster. or Sovereign (Eng.)	$4.86\frac{6}{10}$	Tael, Japan,	$0.75
Pound Ster. Brit. Prov., Canada, Nova Scotia, N. Bruns., Newf'dl'd,	$4.00	Rix Dollar, or Thaler, of Prussia, and Northern German States,	$0.69
Dollar, Mexico, Peru, Chili, Central America,	$1.00	Rouble, silver, Russia,	$0.75
Specie Dollar, Denmark,	$1.05	Guilder, Florin, Netherlands,	$0.40
Specie Dollar, Sweden and Norway,	$1.06	Florin, South Germany,	$0.40
Millree, Portugal,	$1.12½	Millree, Madeira,	$1.00
Pagoda, India,	$1.84	Franc, France, Belgium,	$0.186
Tael, China,	$1.48	Ducat, Naples,	$0.80
Rupee, Bengal,	$0.50	Real of Plate, Spain,	$0.10
Rupee, British India,	$0.41½	Lira, Tuscany,	$0.16
Rix Dollar, or Thaler, Bremen,	$0.78¾	Ounce, Sicily,	$2.40
Pistole, Spain,	$3.904	Doubloon, Mexico,	$15.60
Piaster, Tunis,	$0.124	Ducat, gold, Austria,	$2.278
		Ducat, Hamburg,	$2.257
		Livre, Genoa,	$0.186
		Lira, Genoa,	$1.86

EXAMPLES FOR PRACTICE.

3. What is the cost in Boston of a draft on Liverpool, Eng., for £325 7s. 9d., at 9% premium? Ans. $1576.32+.

NOTE.—Express the face in £, reduce to $ at the rate of £9 = $40.

4. What is the cost in London of a draft on New York for $12000, at 10% premium? Ans. £2454 10s. $10\frac{10}{11}$d.

5. What cost in Charleston a draft on St. Petersburg for 4000 roubles at $1\frac{1}{2}$% premium? Ans. $3045.

6. What cost in Portland a draft on Paris for 10000 francs?

7. What cost a draft on Madrid for 2000 pistoles, at 1 pistole = $4.10?

8. What cost in Pekin a draft on Boston for $5000, exchange 1 tael = $1.62½? What is the % of exchange?

9. What cost in Paris a draft for \$6000, exchange 1 franc = \$.205?

10. A merchant in Bremen wishes to purchase a draft on New York for ten \$1000 5–20 bonds at $115\frac{1}{8}\%$; how much must he pay, exchange being 1 thaler = $\$.86\frac{2}{5}$?

Ans. $\$13333\frac{1}{3}$ thalers.

11. What cost in Pittsburgh a draft on Hamburg for 1800 ducats, exchange 1 ducat = \$2.55?

12. What cost in Calcutta a draft on New Orleans for \$8000, exchange 1 pagoda = \$2.05?

13. What cost a draft on Genoa for 1200 lira, at 10% premium?

ARBITRATION OF EXCHANGE.

Art. 411. Arbitration of exchange is computing exchange between two places by means of intermediate exchanges. Its purpose is to ascertain the cheapest or best route of exchange.

Art. 412. To arbitrate exchange.

Ex. 1. A, in New York, is to pay £2000 in London. If he buys exchange directly on London, it costs 10%: if he buys on Paris, he gets 5 francs 40 centimes for \$1, and the credit thus obtained in Paris can buy exchange on London at £1 for $25\frac{1}{2}$ francs: how much cheaper is payment through Paris than directly? Ans. $\$333\frac{1}{3}$.

ANALYSIS.—£2000 is worth 2000 times $25\frac{1}{2}$ francs in Paris, or 51000 francs, which is worth in New York $\frac{1}{5.40}$ as many dollars, or \$9444.44+. At 10% premium, £1 is worth 1.10 of $\$4.44\frac{4}{9}$, or \$4.8888+, and £2000 is worth 2000 times \$4.8888+, or \$9777.77+. Hence exchange through Paris is cheaper by \$9777.77 — \$9444.44+, or \$333.33+.

PROCESS INDICATED.—$2000 \times \frac{25\frac{1}{2}}{1} \times \frac{1}{5.40} = 9444.44+.$

Rule.—*Multiply the given sum by the ratios of the intervening exchanges.*

EXAMPLES FOR PRACTICE.

2. A New York merchant wishes to pay in London £375, direct exchange on London being 9% premium. Instead of remitting directly, he remitted through Naples and Paris, how much money must he send, if £1 = 6.05 ducats, 5 ducats = 23 francs, and 50 francs = \$9.10? How much would he have saved by remitting directly to London?

Ans. \$1899.39$\frac{3}{4}$. Gain \$82.73$\frac{1}{12}$.

3. A Chicago merchant remits \$4360 to Genoa through London and Paris; what is the value received, if exchange on London is 9% premium, 25.20 francs = £1, and 1 lira = 10 francs? What would he receive by drawing directly on Genoa, at 10% premium?

Ans. Circular, 2268 lira; direct, 2130.98+lira.

4. A merchant of St. Louis has \$12000 to pay in New York. The direct exchange is 1% premium; but exchange on Pittsburgh is $\frac{1}{2}$% premium, and from Pittsburgh to New York is 1% discount. How much will pay his debt by circular exchange, and what does he gain?

Ans. \$11939.40; \$180.60 gain.

5. If 8 horses are worth 12 mules, 20 mules worth 40 cows, 24 cows worth 80 hogs, 12 hogs worth 30 sheep, and 18 sheep worth \$108, how many dollars are 20 horses worth?

Ans. \$3000.

6. A gentleman in Washington drew on Stuttgart, South Germany, for 8120 florins at \$.41 per florin; how much more would he have received if he had ordered remittance to London, and thence to Washington, exchange at Stuttgart on London being 11.7 florins = £1, and at London on Washington 10%, in favor of sterling, brokerage at 1$\frac{1}{2}$% in London for remitting? Ans. \$13.63.

7. A Chicago merchant wishes to remit \$4800 to Boston. Exchange on Boston is $\frac{7}{8}$% premium; but, on Buffalo it is $\frac{1}{2}$% premium; from Buffalo to New York, $\frac{1}{2}$% discount; from New York to Boston, 1$\frac{1}{8}$% premium. What will be the value in Boston by each method, and how much better is the *circular?*

8. If 18 men can earn as much as 24 women, 28 women as much as 42 boys, 25 boys as much as 30 girls, and 11 girls can earn $13.75 in a day, how many dollars can 13 men earn in a day?

EQUATION OF PAYMENTS.

Art. 413. **Equation of payments** is the process of finding that time for the payment of two or more debts between two parties which gives each party his just amount of use of the money.

The **equated,** or **equitable, time** of the debts between two parties is either that date of payment of their sum which gives each party his just amount of use of the money, or that date from which interest should be reckoned on their sum to the time of its payment.

A **focal date** is a date assumed, in equating payments, for the purpose of comparing the distance from it of the dates of an account.

The **time to run,** or **term of credit,** of a payment, is the time between an assumed date and the date of the payment's becoming due.

CALCULATIONS IN EQUATION OF PAYMENTS.

Case I.

Art. 414. To equate the debts of one party to another.

Ex. 1. If you owe me $100 due 6 months hence, and $100 due 12 months hence, when can you make both payments at once so as to give both parties their just amount of use of the money? Ans. 9 months hence.

Process Indicated.

$$\begin{array}{lr} 100 \times 6 = & 600 \\ \underline{100} \times 12 = & \underline{1200} \\ 200 &)\underline{1800} \end{array}$$

Average term of credit, 9 mo.

Explanation.

You should have the use of $100 for 6 months, equivalent to the use of $1 for 100 times 6 mo., or 600 mo.: also the use of $100 for 12 months, equivalent to the use of $1 for 100 times 12 mo., or 1200 mo. Both are equivalent to the use of $1 for 1800 mo., or to the use of $200 for $\frac{1}{200}$ of 1800 mo., that is, 9 mo.

NOTE.—In this case, if you keep the $100 first due 3 months past maturity, you balance your obligation to me by depriving yourself of, and giving me, the use of the second $100 three months before maturity.

Ex. 2. If your purchases of Sims & Sons are April 10, mdse. $1500, May 15, mdse. $2000, and June 20, mdse. $2500, each on a credit of 6 months, when could you pay $6000, the principal of these debts, so as to satisfy all the interest claims of Sims & Sons, and give yourself, the just amount of use of your money? Ans. November 21.

EXPLANATION.—The required date must be such that the amount of use of money kept by you past maturity shall equal the amount of use which you resign to Sims & Sons by paying before maturity. To find this date, we assume a focal date, and determine the time of run, or term of credit, of the debts from that date. We can assume any date whatever, but it is generally most convenient to take as a focal date one of the dates of maturity, such as the first, or the last.

PROCESSES INDICATED.

First Focal Date, Oct. 10.				Second Focal Date, Dec. 20.		
Due.	$.	days.		$.	days.	
Oct. 10,	1500 ×	0 =	0	1500 ×	71 =	106500
Nov. 15,	2000 ×	36 =	72000	2000 ×	35 =	70000
Dec. 20,	2500 ×	71 =	177500	2500 ×	0 =	0
	6000		)249500	6000		)176500
Av. time to run after Oct. 10, $41\frac{7}{12}$ da., gives Nov. 21.			$41\frac{7}{12}$da.	Av. time to run before Dec. 20, $29\frac{5}{12}$ da., gives Nov. 21.		$29\frac{5}{12}$da.

PROOF BY INTEREST.

Int. 6% of $1500 fr. Oct. 10 to Nov. 21, ($41\frac{7}{12}$da.) = $10.395.
" " " $2000 fr. Nov. 15 to " " ($5\frac{7}{12}$da.) = 1.8611+.
Int. $2500 fr. Nov. 21 to Dec. 20, ($29\frac{5}{12}$ da.) = $12.256+.

This proof shows that if by paying all the debts in one payment on Nov. 21, you kept from Sims & Sons $12.256 worth of use of the money past due, you balanced the favor by resigning to them $12.256 worth of use of money before it was due them. The same equation of favors would occur with any other rate of interest than 6%.

Rule.—*Compute the interest on each debt from the date of its maturity to that of the last maturity, and divide the sum of these interests by the interest on the sum of the debts for one of the periods of time. Reckon the quotient back from the last maturity.* Or,

Assume for a focal date that of the first or last maturity, and multiply each debt by the units of time between that date and that of its own maturity; then divide the sum of the products by the sum of the debts, and reckon the quotient from the focal date toward the other dates.

NOTE 1.—Any date may be taken as a focal date, but that of the first or last maturity is generally the most convenient for a finished account. In an account current, or unfinished account, when the book-keeper knows that it must be equated at a particular time, for example the end of the year, he finds it convenient to assume that date as the focal date; and to determine the product at the time of the transaction, which saves labor at settlement. *If a focal date is assumed between the first and last maturity, then the sum of the products on one side of it are to be subtracted from the sum of the products on the other side of it, the difference is to be divided by the sum of the debts, and the quotient reckoned from the focal date on the side of the greatest sum of products.* For example, if, in Ex. 2, we assume Nov. 15 as the focal date, the product before it is 54000, the product after it is 87500, the difference is 33500, which, divided by 6000, gives $5\frac{7}{12}$ days to be reckoned after Nov. 15, because the greatest product is after it. This gives Nov. 21, as in the case of other focal dates.

NOTE 2.—In reckoning the quotient from a focal date, the starting-point is the *end* of the day, and, if the quotient has a fraction, it indicates that the whole day in which that fraction ends is the equated time. This is on the principle that the law takes no note of fractions of days in interest, but the debtor has till the last moment of the business hours of the day of maturity to settle.

NOTE 3.—It is plain that any units of time may be used, but days are the most satisfactory for ordinary transactions.

NOTE 4.—If the equated time comes before one or more of the transactions, thus making settlement at that time impossible, interest should be charged on the sum of the debts from that time till the actual settlement. (See Ex. 6.)

EXAMPLES FOR PRACTICE.

3. I owe A $1600 due in 5 mo., $500 due in 8 mo., and $2400 due in 10 mo. If I give my note for the whole amount, how long should it run? Ans. 8 mo.

4. I buy of a wholesale dealer, at 2 mo. credit, as follows:— Apr. 10, an invoice of $300; Apr. 25, $400; May 1, $100; May 15, $500. What is the average time for the payment of the whole? Ans. June 30.

5. Bought goods of F. S. Stewart & Co., as follows:—

1872.	Jan. 15,	To mdse.	on	3 mo.	credit,	$350.
"	Feb. 20,	"	"	4 "	"	$400.
"	Mar. 10,	"	"	5 "	"	$500.
"	Apr. 10,	"	"	6 "	"	$700.
"	May 1,	"	"	30 da.	"	$200.
"	June 12,	"	"	60 "	"	$300.

What is the equated time of payment of this bill?
Ans. July 28, 1872.

6. I sell goods to A. Bates & Co. at different times, and for different terms of credit, as follows:—

1871.	Aug. 21,	To mdse.	on	2 mo.	credit,	$600.
"	Oct. 12,	"	"	3 "	"	$400.
"	Dec. 18,	"	"	60 da.	"	$150.
1872.	Jan. 15,	"	"	30 "	"	$1000.
"	Feb. 20,	"	"	90 "	"	$250.

If I take their note in settlement, at what time should interest commence? Ans. Jan. 21, 1872.

7. Bought goods of Grant, Wilson & Co., as follows:—

1869.	Sept. 15,	Mdse.,	on	60 da.	credit,	$720.
"	Dec. 10,	"	"	30 "	"	$300.
1870.	Feb. 12,	"	"	90 "	"	$100.
"	Mar. 25,	"	"	1 mo.	"	$500.
"	May 2,	"	"	30 da.	"	$450.
"	May 15,	"	"	10 "	"	$1200.

What is the equated time of payment?

CASE II.

Art. 415. To equate the debts of two parties to each other.

Ex. 1. Find the equated time for paying the balance of the following account. Ans. Aug. 7.

Dr. A IN ACC'T CURRENT AND INT. ACC'T WITH B. Cr.

1871.		$	ct.	1871.		$	ct.
Jan. 3.	To Mdse. (6 mo. cr.)	1500		Feb. 21.	By Mdse. (5 mo. cr.)	1800	
Feb. 14.	" " (5 mo. cr.)	2500		Mar. 28.	" " (4 mo. cr.)	3000	
Mar. 21.	" " (4 mo. cr.)	7200		May 10.	" Cash.	2000	
Apr. 12.	" Cash.	500		June 14.	" "	1000	

EXPLANATION.—In this account

A owes B	B owes A
$1500 due July 3.	$1800 due July 21.
$2500 " " 14.	$3000 " " 28.
$7200 " " 21.	$2000 " May 10.
$ 500 " Apr. 12.	$1000 " June 14.

If we assume for a focal date April 12, the date of the earliest maturity, and from it equate each party's debts separately, we have

A's debts equivalent to the use of $1, after focal date, Days.

1500 ×	82 =		123000
2500 ×	93 =		232500
7200 ×	100 =		720000
500 ×	0 =		0
11700			1075500
7800			620000
3900		)	455500

B's debts equivalent to the use of $1, after focal date, Days.

1800 ×	100 =		180000
3000 ×	107 =		321000
2000 ×	28 =		56000
1000 ×	63 =		63000
7800			620000

NOTE.—A owes $3900 more than B, and his indebtedness is equivalent to the use of $1 for 455500 more days than B's, or to the use of $3900 for $\frac{1}{3900}$ of 455500 days, namely, $116\frac{31}{39}$ days more than B. Hence A owes B $3900, due $116\frac{31}{39}$ days after April 12, viz., Aug. 7.

If we assume for a focal date July 28, the date of the latest maturity, and equate each party's debts separately, it will appear that B then owes *for the use of* \$1 for 38200 days more than A. But A *actually* owes \$3900 more than B. Hence to balance B's debt for use, A should keep his \$3900 for $\frac{1}{3900}$ of 38200 days, or $9\frac{31}{39}$ days after July 28, that is, till Aug. 7.

NOTE.—The above methods are as just as would be the actual computation of use by interest. In fact, these methods are interest *with the common factor, rate per cent., left out.* By the interest method, the balance of interest, divided by the interest of the balance of items for 1 day, gives the number of days from focal date. (See Ex. 2.)

Rule.—*Assume for a focal date that of the first or last maturity, or a date before the first, or after the last. Find the interest of each debt for the time between its maturity and the focal date. Divide the balance of interest by the interest of the balance of items for one of the periods of time. Reckon the quotient* TOWARD *the other dates when both balances are on the same side of the account, but* FROM *the other dates when the balances are on different sides of the account.* Or,

Multiply each debt by the units of time between its maturity and such focal date, and divide the balance of products by the balance of items. The quotient is the number of periods of time to be reckoned TOWARD *the other dates when both balances are on the same side of the account, but* FROM *the other dates when the balances are on different sides of the account.*

EXAMPLES FOR PRACTICE.

2. Solve Example 1 by the first rule, namely, by interest, assuming Apr. 12 for focal date, and rate 6%.

Ans. Bal. of int. \$75.91$\frac{2}{3}$: int. of \$3900 for 1 day \$.65: quotient $116\frac{31}{39}$ da.

ANALYSIS.—If A's interest debt from focal date is \$75.91$\frac{2}{3}$ more than B's, and the interest of his excess of debt is for 1 day 65 cents, then it will take as many days for the interest of his excess of debt to equal his excess of interest as 65 cents is contained times in \$75.91$\frac{2}{3}$, namely, $116\frac{31}{39}$ days.

3. Find the equated time for paying the balance of the following account.

Dr. C IN ACC'T CURRENT AND INTEREST ACC'T WITH D. *Cr.*

1873.		$	cts.	1873.		$	cts.
Feb. 10	To M'dse. (2 mo. cr.)	600	00	Feb. 20	By M'dse. (60 da. cr.)	400	00
Mar. 15	" " (3 mo. cr.)	400	00	Mar. 10	" " (2 mo. cr.)	250	00
May 20	" " (30 da. cr.)	800	00	May 15	" Cash	500	00
July 10	" Cash	300	00	June 20	" Cash	700	00

Ans. Aug. 11.

4. What is the balance of the following account, and when is it due?

Dr. JOHN DAVIES. *Cr.*

1870.		$	cts.	1870.		$	cts.
Sep. 20	To M'dse. (3 mo. cr.)	200	00	Oct. 20	By M'dse. (2 mo. cr.)	300	00
Oct. 12	" " (2 mo. cr.)	500	00	Dec. 10	" Cash	100	00
Dec. 25	" " (10 da. cr.)	600	00	Jan. 15	" Cash	700	00

Ans. Balance $200; due Oct. 20, 1870.
Due Dec. 10, 1870, the amount of $200 for 51 days.

5. Required the balance of the following account and when it is due.

Dr. U. S. GRANT. *Cr.*

1871.		$	cts.	1871.		$	cts.
April 1	To M'dse. (2 mo. cr.)	325	00	Apr. 25	By M'dse.	250	00
June 4	" Sundries, (30 d. cr.)	450	00	May 25	" Sundries	300	00
July 10	" M'dse.	625	00	July 25	" Cash	750	00

Ans. Bal. $100; due Sep. 10, 1871.

6. When shall a draft for the settlement of the following account be made payable, and what the face?

Dr. CHAS. HARRISON. *Cr.*

1873.		$	cts.	1873.		$	cts.
Jan. 6	To M'dse.	324	25	Jan. 30	By Draft	200	00
Mar. 10	" Sundries	112	50	Mar. 15	" Cash	225	00
Apr. 15	" M'dse	50	25	May 16	" M'dse.	35	50

BALANCING INTEREST ACCOUNTS.

Art. 415. To find the balance of an account current at settlement. (See Art. 414.)

Rules.—I. By Equating. *Equate the account. If settlement is at the equated time, the balance of items is the true balance. If settlement is before the equated time, deduct from the balance of items its interest for the intervening time. If settlement is after the equated time, add to the balance of items its interest for the intervening time.*

II. By interest. *Find the interest of each item from maturity to settlement. To each item which matures before settlement add its interest. From each item which matures after settlement subtract its interest. Find the difference of the two sides, after these changes.* Or

Multiply each item by the units of time from maturity to settlement, adding to the products of the other side the product of each item which is paid before maturity. Multiply the difference of the sums of these products by the interest of one unit of money for one unit of time. Add the product to the items of that side which produces the greater sum of products, then find the difference of the sides.

III. By ready balances. *At every transaction multiply the balance then existing by the number of days it has remained unchanged. At settlement add to the side which produced the greatest sum of products the interest on the excess of its products for one day, then find the difference of the sides.*

Examples for Practice.

Ex. 1. Find the balance of the following account at 6% interest, July 1, 1871, by each method. Ans. $2165.51.

1. By Equating. Focal Date at Settlement.

A. IN ACC'T CUR. AND INT. ACC'T WITH B. TO JULY 1, 1871.

Dr.

DATE.	DUE.		$	Cts.	Days.	Product.
1871.	1871.					
Jan. 2	Jan. 2	To bal. due	875		180	157500
Feb. 10	May 10	" M'dse. @ 3 mo.	1550	50	52	80626
Mar. 25	May 25	" " @ 2 mo.	975	75	37	36103
May 25	Aug. 25	" " @ 3 mo.	1248	25		
		" *dis. on $1000, per contra*			4	4000
			4649	50		278229
			2500			182154
			2149	50		) 96075

Cr.

DATE.	DUE.		$	Cts.	Days.	Product.
1871.	1871.					
Feb. 2	Feb. 2	By cash	500		149	74500
May 10	May 23	" draft @ 10 da.	1000		39	39000
June 2	July 5	" note @ 30 da.	1000			
		" *dis. on $1248.25 per contra*			55	68654
			2500			182154

Equated time before focal date, 44.69+ days, or May 18, 1871. Int. @ 6% on \$2149.50 for 44.69 da., \$16.01 : \$2149.50 + \$16.01 = \$2165.51, Ans.

2. By Interest of Each Item.

A. in acc't cur. and int. acc't with B. to July 1, 1871.

Dr.

DATE.	DUE.		$	Cts.	Days.	Interest. $	Interest. Cts.
1871.	1871.						
Jan. 2	Jan. 2	To bal. due	875		180	26	25
Feb. 10	May 10	" M'dse. @ 3 mo.	1550	50	52	13	44
Mar. 25	May 25	" " @ 2 mo.	975	75	37	6	01
May 25	Aug. 25	" " @ 3 mo.	1248	25			
			4649	50		45	70
			45	70			
			4695	20			
		Less discount on $1248.25	11	44			
			4683	76			
			2518	25			
			2165	51			

Cr.

DATE.	DUE.		$	Cts.	Days.	Interest. $	Interest. Cts.
1871.	1871.						
Feb. 2	Feb. 2	By cash	500		149	12	42
May 10	May 23	" draft @ 10 days	1000		39	6	50
Jan. 2	July 5	" note @ 30 days	1000				
			2500			18	92
			18	92			
			2518	92			
		Less discount on $1000		67			
			2518	25			

3. By Ready Balances.

A. IN ACC'T CUR. AND INT. ACC'T WITH B. TO JULY 1, 1871.

Date.	Dr. Items. $	Cts.	Cr. $	Cts.	Dr. Item. Balances. $	Cts.	Cr. $	Cts.	Days	Product. $	Cts.	Balances. $	Cts.
1871.													
Jan. 2	875				875				31	27125			
Feb. 2			500		375				97	36375			
May 10	1550	50			1925	50			13	25031	50		
May 23			1000		925	50			2	1851			
May 25	975	75			1901	25			37	70346	25		
July 5			1000				1000		4	4000			
Aug.25	1248	25			1248	25			55			68653	75
	4649	50	2500							164728	75	68653	75
	16	01								68653	75		
	4665	51								6)96075			
	2500												
	2165	51 =Cash. bal.								16.012=Int. for 1 day.			

Explanation.—In the first method, A's uses of B's money on July 1, are equivalent to the use of \$1 for 157500+80626+36103 days, + his use of A's note of \$1000 for 4 days, equal to \$1 for 4000 days, (B losing and A gaining its discount for 4 days.) In all, A's uses of B's money equal \$1 for 278229 days. On July 1, B's uses of A's money are equivalent to the use of \$1 for 74500+39000 days, + his use of A's debt of \$1248.25 paid 55 days before due, equal to \$1 for 68654 days. In all, B's uses of A's money equal \$1 for 182154 days. Hence A's uses, more than B's, equal \$1 for 96075 days, or \$2149.50 for $\frac{1}{2149.5}$ of 96075 days, that is 44.69+days, which makes May 18 the equated time, and the cash balance, July 1, is the balance of items, \$2149.50,+ its interest for the days between focal date and settlement, \$16.01, namely, \$2165.51.

In the second method, the pupil can, by ruling new columns for each side, fill those columns with the items plus their interest, or less their discount, as the case may be, and then find the balance of the sides. For lack of room for this, the processes are performed below the form.

In the third method, the balance between any two consecutive dates is multiplied by its time of running. Thus, A's use of \$875 was for 31 days, equal to \$1 for 27125 days. Then, paying \$500, he used \$875—\$500=\$375 for 97 days, equal to \$1 for 36375 days; and so on till July 1. Then the items not matured are discounted as in the first method. A's excess of use, \$1 for 96075 days, is equal to $\frac{1}{6}$ as many mills, or \$16.01. This added to his items of debt, makes the total 4665.51, &c.

2. Find the cash balance of the following account, Sep. 1, 1870, at 6%.

Dr. G. WARD IN ACCOUNT WITH F. ROE. *Cr.*

1870.		$	Cts.	1870.		$	Cts.
Jan. 8	To M'dse. (6 mo. cr.)	315	00	Jan. 12	By M'dse. (6 mo. cr.)	250	00
Feb. 10	" " (4 mo. cr.)	400	00	Mar. 1	" " (6 mo. cr.)	430	00
Mar. 5	" " (2 mo. cr.)	500	00	Mar. 25	" Cash	500	00
Apr. 10	" " (3 mo. cr.)	600	00	Apr. 15	" Cash	100	00

Ans. $540.88+.

3. What is the balance of the following account, Jan. 10, 1873, at 8%?

Dr. S. KERR IN ACCOUNT WITH M. KING. *Cr.*

1872.		$	Cts.	1872.		$	Cts.
May 8	To M'dse. (6 mo. cr.)	265	40	June 18	By M'dse. (4 mo. cr.)	300	00
July 9	" " (3 mo. cr.)	175	60	Aug. 12	" " (2 mo. cr.)	87	50
Sep. 24	" " (30 da. cr.)	215	00	Oct. 8	" Cash	200	00

4. What is the balance of the following account, Oct. 1, 1872, at 6%?

Dr. J. FRASER IN ACCOUNT WITH D. S. NOBLE. *Cr.*

1872.		$	Cts.	1872.		$	Cts.
Jan. 15	To M'dse. (6 mo. cr.)	1500	00	Mar. 10	By M'dse. (6 mo. cr.)	1000	00
May 10	" " (3 mo. cr.)	2000	00	May 15	" " (2 mo. cr.)	2100	00
July 5	" " (2 mo. cr.)	3000	00	Aug. 1	" Draft @ 15 da.	2000	00
Sep. 15	" " (2 mo. cr.)	2000	00	Oct. 26	" Cash	3000	00

Ans. $1302.199+.

5. Balance the following account, at 6%, May 1, 1871.

Dr. WM. BRUNT, JR., IN ACCOUNT WITH S. SIMMS. *Cr.*

1870.		$	Cts.	1870.		$	Cts.
Sep. 15	To M'dse. (6 mo. cr.)	455	00	Nov. 10	By M'dse. (6 mo. cr.)	325	80
Oct. 10	" " (6 mo. cr.)	460	60	Dec. 10	" Note, at 3 mo.	550	00
Dec. 21	" " (3 mo. cr.)	508	40	Jan. 15	" Draft, at 10 da.	465	20

SYNOPSIS OF THE APPLICATIONS OF PERCENTAGE.

	APPLICATIONS.
Percentage.	Partnership.
	Bankruptcy.
	Taxes.
	Insurance.
	Commission.
	Profit and Loss.
	Interest.
	Discount.
	Stocks.
	Exchange.
	Accounts.

CHAPTER XVII.

APPLICATIONS OF AVERAGE.

Art. 416. Averaging is in general use as a means of dealing with unequal quantities related to a single purpose or result. (See Articles 91 and 92.) Its applications aim at a correct estimate of totals, and a just apportionment of benefits and burdens. Its most prominent commercial applications are in estimates of allowances and net results on large lots of separate pieces of merchandise, property, causes, effects, accidents, &c., in the assessment of the benefits, contributions, losses, and taxes, which concern partnerships, governments, public improvements, damages by natural causes, &c., and in the determination of the values and qualities of compounds from their ingredients, or of ingredients for a given compound. Some of these applications have been discussed in the foregoing pages, and those only will be discussed in this chapter which relate to *rate bills* and *compounds.*

RATE BILLS.

Art. 417. A **rate bill** is a statement of the rates or quantities accruing to parties from their relation to a certain purpose or result.

Contributory interests are the values concerned in a certain purpose or result.

It is generally believed to be just that rates of benefit or burden should be in proportion to the contributory interests. There is, therefore, the only rule, in substance, for determining all such cases.

Rule.—*First ascertain the contributory interests, then divide the burden or benefit among them in proportion to their values.*

NOTE.—Statute law, custom, or the courts, sometimes define the proportions which shall govern distribution of prize money, rewards, &c. These cases rest more on personal and official merit than upon a money value of contributory interests.

Art. 418. In marine ventures, averaging is used to adjust, among the contributory interests of a voyage, any partial losses which are voluntarily suffered to save the rest, such as throwing overboard a part of the cargo to relieve the distress of the vessel, &c.

General average, in marine ventures, is an apportionment, among contributory interests, of losses voluntarily suffered when in imminent danger, so as to save the rest.

Jettison is the portion of goods thrown overboard. The word is from the French *jeter,* to throw.

Salvage is the pay of those whose exertions have rescued vessels or goods from imminent danger.

Particular average is an adjustment of indemnity for damage not voluntarily suffered. It is based on a comparison of the gross proceeds of damaged goods and sound goods.

Practice is generally according to the following points:—

1. The ship, the freight charges, and everything on board which pays freight, are contributory interests.

2. Seamen's wages are not contributory interests, that seamen may have no interest in opposing the sacrifice.

3. The cargo, including the part sacrificed, is valued at what it would have sold for in its destined market.

4. One-third is usually deducted from the cost of ship repairs, since the new is considered better than the old.

5. One-third, (in New York $\frac{1}{2}$,) is usually deducted from the freight charges for seamen's wages, pilotage, &c. The remainder is the contributory interest of the freight.

6. Insured parties recover losses incurred by general average.

Rate Bill Examples for Practice.

1. A school pays $900 for teacher's salary, $89.20 for repairs, $52.80 for fuel, and $10 for incidentals, and draws $560 public money, the balance to be paid by the patrons; the whole number of days of attendance is 5600. Find the rate per day, and the bill of A, who sends one pupil 120 days; of B, who sends 2 pupils 115 days each, of C, who sends one pupil 124 days, one 86 days, and one 40 days.

Ans. Rate, $8\frac{11}{14}$ cts.; A pays 10.54\frac{2}{7}$; B, 20.20\frac{5}{7}$; C, 21.96\frac{3}{7}$.

2. The ship Ocean, overtaken by a storm, threw overboard goods worth $15000, and put into port for repairs. The expenses of detention were $300; repairs $1800. Divide the less, the vessel being worth $40000, the freight $4500, the cargo $41000. A's interest is $6000, B's $10000, C's $12000, D's $8000, E's $5000. The property lost was $3000 of A's, $4000 of B's, and $8000 of C's.

Ans. Rate of loss, $\frac{11}{56}$; ship owners pay 8446.42\frac{6}{7}$, and receive $1500, making a balance paid, 6946.42\frac{6}{7}$; A pays 1178.57\frac{1}{7}$, and receives $3000, making a balance received of 1821.42\frac{6}{7}$; B pays 1964.28\frac{4}{7}$, and receives $4000, balance received 2035.71\frac{3}{7}$; C pays 2357.14\frac{2}{7}$, and receives $8000, balance received 5642.85\frac{5}{7}$; D pays 1571.42\frac{6}{7}$; E pays 982.14\frac{2}{7}$.

COMPOUNDS.

Art. 419. Averaging, when applied to compounds, has for its purposes the determination of the value of a unit of the whole from the numbers and values of the constituents, or the determination of the proportions which correspond to certain values. When thus applied, the processes have been called *Medial Proportion*, in reference to the principle of mean or average, and *Alligation*, in reference to the methods of representing the quantities as *bound together by lines* so as to represent comparison to the eye.

Alligation is a method of solving questions of average by representing the compared quantities as connected by lines. The name is derived from the Latin *alligo*, to bind or *tie to* anything.

CASE I.

Art. 420. To find the value of a unit of the compound, when the constituents and their values are known.

Ex. 1. Bought 12 lb. tea @ 75 ct., and 16 lb. @ \$1. Find the average cost per lb. Ans. \$0.89$\frac{2}{7}$.

WRITTEN PROCESS.

$$\begin{array}{ll} 12 \times \$0.75 = & \$\ 9.00 \\ \underline{16 \times \$1.} \quad = & \underline{\$16.00} \\ 28 &) \$25.00 (\$.89\tfrac{2}{7} \end{array}$$

EXPLANATION.

Twelve lb. at 75 ct. cost \$9, and 16 lb. at \$1 cost \$16. Therefore 12 lb. + 16 lb., or 28 lb., cost \$9 + \$16, or \$25, and 1 lb. costs $\frac{1}{28}$ of \$25, or 89$\frac{2}{7}$ ct.

Rule.—*Find the value of each constituent, and divide the sum of their values by the sum of the constituents.*

EXAMPLES FOR PRACTICE.

2. If I mix 6 lb. of sugar at 10 cts. a pound, and 5 lb. at 12 cts. a pound with 9 lb. at 16 cts. a pound, what is a pound of the mixture worth? Ans. \$.13$\frac{1}{5}$.

3. A grocer mixed 8 lb of tea at \$.62$\frac{1}{2}$ with 6 lb. at \$.75, 3 lb. at \$1, and 8 lb. at \$1.25; how much is the mixture worth per pound?

4. A merchant bought 108 yd. of cloth at \$4.50 per yard, 24 yd. at \$5.25, and 36 yd. at 6$\frac{1}{3}$: what was the average cost per yard?

5. What per cent. of alcohol in a mixture of 7 gal., 90% strong; 8 gal., 95% strong; 10 gal., 99% strong; and 3 gal. of water? Ans. 85%.

6. A merchant bought 2000 lb. of wool at \$.47 per pound, 3000 lb. at \$.44, 5000 lb. at \$.48, and 8000 lb. at \$.45; at what average price per pound must he sell it to gain 5%?

7. If 3 oz. of gold 22 carats fine be compounded with 6 oz. 23 carats fine, 8 oz. 22$\frac{1}{2}$ carats fine, 5 oz. 20 carats fine, and 2 oz. of alloy, how many carats fine is the composition? Ans. 20$\frac{1}{8}$.

CASE II.

Art. 421. To find the proportions of constituents of known value, to make a compound of a given value.

Ex. 1. What proportion of simples, costing, respectively, 15 ct. and 24 ct., make a unit of the compound cost 18 ct.?

Ans. Of the 15 ct., $\frac{2}{3}$ of the compound; of the 24 ct., $\frac{1}{3}$.

FIRST METHOD.

$$18\left\{\begin{array}{c|c|c} 15 & 6 & 2 \\ 24 & \underline{3} & \underline{1} \\ & 9 & 3 \end{array}\right.$$

SECOND METHOD.

$$18\left\{\begin{array}{c|c|c|c} 15 & \frac{1}{3} & \frac{2}{6} & 2 \\ 24 & \frac{1}{6} & \frac{1}{6} & \underline{1} \\ & & & 3 \end{array}\right.$$

ANALYSIS OF FIRST METHOD.

Since every unit of the 15 ct. simple, when sold at 18 ct., gains 3 ct., and since every unit of the 24 ct. simple, when sold at 18 ct., loses 6 ct., it will take as many of the 15 ct. simple to gain what one of the 24 ct. simple loses as 3 is contained times in 6, that is, 2 of the 15 ct. simple. Therefore the 15 ct. is 2 parts, the 24 ct. is 1 part, the whole is 3 parts, the 15 ct. is $\frac{2}{3}$ of the whole, and the 24 ct. is $\frac{1}{3}$ of the whole.

ANALYSIS OF THE SECOND METHOD.—Since every unit of the 15 ct. simple, when sold at 18 ct., gains 3 ct., it takes $\frac{1}{3}$ of such unit to gain 1 ct. Since every unit of the 24 ct. simple, when sold at 18 ct., loses 6 ct., it takes $\frac{1}{6}$ of such unit to lose 1 ct. Therefore, as the gain and loss are equal, the quantities of the constituents are as $\frac{1}{3}$ to $\frac{1}{6}$, or as $\frac{2}{6}$ to $\frac{1}{6}$, or as 2 to 1. Therefore the 15 ct. is $\frac{2}{3}$ of the whole, and the 24 ct. is $\frac{1}{3}$ of the whole.

PROOF.—$2 \times 15 = 30$: $1 \times 24 = 24$: $(30 + 24) \div (2 + 1) = 18$.

Ex. 2. What proportions of constituents, at 10, 12, 15, and 18 cents, respectively, make a unit of the compound cost 14 cents?

SOME ANSWERS.

1. The 10 and 12 each $\frac{5}{22}$; the 15 and 18 each $\frac{6}{22}$ of the com.
2. The 10 and 18 each $\frac{4}{11}$; the 12 ct. $\frac{1}{11}$, the 15 ct. $\frac{2}{11}$ of the com.
3. The 12 and 15 each $\frac{4}{11}$; the 10 ct. $\frac{1}{11}$, the 18 ct. $\frac{2}{11}$ of the com.
4. The 12 and 15 each $\frac{4}{19}$; the 10 ct. $\frac{5}{19}$, the 18 ct. $\frac{6}{19}$ of the com.
5. The 10 ct. $\frac{1}{14}$, 12 ct. $\frac{5}{14}$; 15 ct. $\frac{6}{14}$, 18 ct. $\frac{2}{14}$ of the com.
6. The 10 ct. $\frac{4}{17}$, 12 ct. $\frac{5}{17}$, 15 ct. $\frac{2}{17}$, 18 ct. $\frac{6}{17}$ of the com.
7. The 10 ct. $\frac{5}{16}$, 12 ct. $\frac{1}{16}$, 15 ct. $\frac{6}{16}$, 18 ct. $\frac{4}{16}$ of the com.

WRITTEN PROCESSES.

1. $14\left\{\begin{array}{l}10\\12\\15\\18\end{array}\right.\quad\begin{array}{r}1+4=5\\1+4=5\\4+2=6\\4+2=\underline{6}\\22\end{array}$
2. $14\left\{\begin{array}{l}10\\12\\15\\18\end{array}\right.\quad\begin{array}{r}4\\1\\2\\\underline{4}\\11\end{array}$
3. $14\left\{\begin{array}{l}10\\12\\15\\18\end{array}\right.\quad\begin{array}{r}1\\4\\4\\\underline{2}\\11\end{array}$

4. $14\left\{\begin{array}{l}10\\12\\15\\18\end{array}\right.\quad\begin{array}{r}1+4=5\\4\\4\\4+2=\underline{6}\\19\end{array}$
5. $14\left\{\begin{array}{l}10\\12\\15\\18\end{array}\right.\quad\begin{array}{r}1\\1+4=5\\4+2=6\\\underline{2}\\14\end{array}$

6. $14\left\{\begin{array}{l}10\\12\\15\\18\end{array}\right.\quad\begin{array}{r}4\\1+4=5\\2\\4+2=\underline{6}\\17\end{array}$
7. $14\left\{\begin{array}{l}10\\12\\15\\18\end{array}\right.\quad\begin{array}{r}1+4=5\\1\\4+2=6\\\underline{4}\\16\end{array}$

EXPLANATION.—The average cost, 14 ct., is placed at the left of the column of prices. A line, drawn from any price lower than the average to a price higher than the average, indicates that those two prices are compared with the average and each other. Seven styles of connecting and comparing the prices are shown. Other styles, made up of the sums or differences of any of these seven, will show other correct proportions. For example, adding the numbers of the last two sets, we have another correct set, viz., 9, 6, 8, and 10 thirty-thirds. In the first written process, 10 is compared with 15 and 18; and 12 is compared with 15 and 18. Now, a simple at 10, sold at 14, gains 4 cents: and a simple at 15, sold at 14, loses 1 cent. Therefore it will take 4 units of the 15 ct. simple for every unit of the 10 ct. simple to keep the average cost 14 cents. Therefore place 4 opposite 15 for its proportion, and 1 opposite 10 for its proportion. Again, comparing the 10 ct. simple with the 18 ct. simple, that at 18 loses 4, while that at 10 gains 4. Therefore 4 of the 18 loses as much as 4 of the 10 gain. Hence place 4 opposite 10 and 18 as the proportion of each. In the same way, comparing 12 with 15 gives 2 for the 15, and 1 for the 12; and comparing 12 with 18 gives 2 for the 18 and 4 for the 12. Therefore the proportions are, for the 10 and 12, each $1 + 4 = 5$, and, for the 15 and 18, each $4 + 2 = 6$. Since the sum of these proportional numbers is 22, the 10 ct. and 12 ct. simples are each $\frac{5}{22}$ of the whole, and the 15 ct. and 18 ct. are each $\frac{6}{22}$ of the whole. Let the learner explain each of the other six methods, and form new proportions by addition and subtraction of the corresponding numbers of the different styles.

Rule.—*Find the gain or loss on one of each constituent, then take as much of each as will make the gain and loss equal.* Or,

Write the values of the constituents in a column on the right of a vertical line, and that of the compound on the left. Draw a line connecting each value less than that of the compound with one or more of those greater.

Write the difference between the value of the compound and that of each constituent opposite every value with which it is connected.

The sum of the differences opposite any value may be taken as the quantity of that constituent. The ratio of this sum to the sum of all the differences is the ratio of that constituent to the whole.

PROOF.—Multiply each quantity by its value, and divide the sum of the products by the sum of the quantities. The quotient should be the value of the mixture.

NOTE 1.—After finding all the ratios developed by all possible methods of connecting and comparing the values, new ratios may be developed by finding the sums or differences of corresponding values in the different sets.

NOTE 2.—Such questions have an unlimited number of particular answers, because each of all the possible ratios may be filled by an unlimited number of supposed numbers.

NOTE 3.—When the value of a constituent is that of the compound, it need not be used in the written process, because any quantity of it may be added to the compound without affecting its value per unit. In such a case, this rule finds the proportions of the others among themselves.

NOTE 4.—Any common factor may be omitted from the proportionals.

NOTE 5.—This case is often called *Alligation Alternate.*

EXAMPLES FOR PRACTICE.

Find seven answers to each of the following:—

3. How much sugar worth 10, 12, and 15 cents a pound, must be mixed together, so that the mixture shall be worth 13 cents a pound? Ans. 2, 2, and 4, or 1, 1, and 2.

4. How many pounds of tea at \$.50, \$.60, \$.80, and \$1.20 a pound, must be mixed together, so that the mixture shall be worth \$.75 a pound? Ans. 10, 9, 5, and 8.

5. What relative quantities of alcohol, 86, 92, 95, and 98% strong, will make a mixture 93% strong?

6. What relative quantities of gold, 16, 18, 19, 22, and 23 carats fine, will make a metal 20 carats fine?

7. Bought cows at \$10 each, hogs at \$3, sheep at \$$\frac{1}{2}$, so that the average cost per head was \$1; how many of each did I buy? Ans. 22 sheep, 1 hog, 1 cow.

CASE III.

Art. 422. To find the other quantities, when that of any constituent, or of the compound, is known.

Ex. 1. In Ex. 2, Case II., suppose 10 lb. of the 10 ct. constituent; find the other quantities.

ANALYSIS BY THE FIRST PROCESS.—If 10 lb. is $\frac{5}{22}$ of the whole, then $\frac{1}{5}$ of 10 lb., or 2 lb., is $\frac{1}{22}$ of the whole, and 22 times 2 lb., or 44 lb., is the whole. Hence the 12 ct. constituent is 10 lb., and the 15 and 18 ct. constituents are each $\frac{6}{22}$ of 44, or 12 lb.

NOTE.—Analyze this supposition by the other processes.

Ex. 2. In Ex. 2, Case II., suppose 88 bushels to be the compound; find the quantities of the constituents.

ANALYSIS BY THE SECOND PROCESS.—If 88 bu. is the whole, then the 10 ct. and 18 ct. constituents are each $\frac{4}{11}$ of 88 bu., or 32 bu., the 12 ct. constituent is $\frac{1}{11}$ of 88 bu., or 8 bu., and the 15 ct. constituent is $\frac{2}{11}$ of 88 bu., or 16 bu.

NOTE.—Analyze this supposition by the other processes.

Rule.—*Find the proportions by Case II., and with these find the required quantities from the given quantity.*

EXAMPLES FOR PRACTICE.

3. How many pounds of teas at 60, 80, and 90 cents a pound, must be mixed with 29 lb. at \$1.25 a pound, so that the mixture may be worth \$.96 a pound? Ans. 14$\frac{1}{2}$ lb.

4. What relative quantities of sugar at 6, 8, 10, and 15 cts. will make a mixture of 66 lb. at 9 cts. a pound?

5. How much alcohol, 75%, 80%, 90%, and 100% strong make 72 gal. 85% strong?

6. A merchant sold salt for \$2, flour for \$9, sugar for \$30, and 40 barrels of molasses for \$40 a barrel; what was the number of barrels of each, if the average price was \$20 a barrel?

7. A farmer sold sheep for \$5, hogs for \$8, cows for \$20, and 26 mules for \$40 each; the average price of all was \$18 each; how many sheep, hogs and cows did he sell?

Ans. 64, 1, 5, or 64, 2, 10, or 64, 3, 15, &c.

8. A merchant bought some hats for \$4 each, some vests for \$6 each, and 24 coats at \$16 each; the average cost of all was \$10; how many hats and vests did he buy?

One answer, 8 hats, 24 vests.

9. A grocer wishes to mix 20 lb. of sugar at 8 cts. and 30 lb. at 10 cts. with some at 15 cts. and 20 cts., so that the mixture may be worth 12 cts.; how much of the latter kind must he take? Ans. 20 lb. at 15 cts.; 10 lb. at 20 cts.

NOTE.—We find by Case II. that the number at 8 and 20 cts. are as 2 to 1, and at 10 and 15 cts. as 3 to 2; hence, as often as we take 2 lb. at 8 cts. we take 1 lb. at 20 cts., and as often as we take 3 lb. at 10 cts. we take 2 lb. at 15 cts.

10. A jeweller has 2 pwt. 15 gr. of gold, 16 carats fine; how much gold, $21\frac{3}{8}$ carats fine, must be mixed with it, to make the mixture 18 carats fine? Ans. 1 pwt. 11 gr.

11. How many pounds of tea at \$.50 and \$.60 per pound must be mixed with 24 lb. at \$.80 and 30 lb. at \$.90, so that it may average \$.75 a pound?

Ans. 18 lb. at \$.50; 8 lb. at \$.60.

12. A's farm cost him, on the average, \$60 an acre. He gave for 100 acres of it \$50 an acre, and for the rest of it \$85 an acre. How many acres in his farm? Ans. 140.

13. A stock-dealer bought 270 head of sheep for \$1365, paying \$4, $\$4\frac{1}{2}$, $\$5\frac{1}{2}$, and $\$7\frac{1}{2}$ a head; how many did he buy *at each* rate? One answer, 40, 110, 95, 25.

14. A boy has some small coins whose denominations are respectively 1 ct., 2 cts., 5 cts., and 10 cts., which he wishes to exchange for 60 three-cent pieces; how many must he exchange of each kind?

15. A farmer bought 100 head of stock for \$100, paying $\$\frac{1}{2}$ each for sheep, \$3 for hogs, and \$10 for cows; how many of each did he buy?

Ans. 94 sheep, 1 hog, 5 cows.

Synopsis of Applications of Average.

Average. { Rate Bills—Marine Average.
Compounds—Alligation.

CHAPTER XVIII.

INVOLUTION. EVOLUTION.

INVOLUTION.

Art. 423. Involution is the process of finding a power of a number.

A **power** of a number is either the number itself, or a product whose component factors are that number. Thus, 32 is a power of 2, being equal to $2 \times 2 \times 2 \times 2 \times 2$; also, 2 is the first power of itself. (See Art. 44.)

Involving a number is using it to produce its power.

A **root** of a given number is either the number itself, or that number whose involution produces the given number. Thus, 2 is a root of 32, because involving 2 produces 32; also, 32 is the first root of itself.

Art. 424. Powers are named from the number of times the root is used in producing them.

The **first power** of a number is the number itself.

The **second power,** or **square,** of a number is the product obtained by using that number twice as a factor. Thus, 49 is the second power, or square, of 7, because $7 \times 7 = 49$.

The **third power,** or **cube,** of a number is the product obtained by using that number three times as a factor. Thus, 27 is the third power, or cube, of 3, because $3 \times 3 \times 3 = 27$.

NOTE.—The second power is called a *square*, because the area of a square is the product of two equal factors. (See Art. 213.) The third power is called a *cube*, because the volume of a cube is the product of *three equal* factors. (See Art. 217.)

A **perfect power** is a number which can be produced by involution. Thus, 16 is a perfect square, and 27 a perfect cube.

An **imperfect power** is a number which cannot be produced by the requisite involution. Thus, 27 is an imperfect square, because it is not the product of two equal factors, and 16 is an imperfect cube, because it is not the product of three equal factors.

The **index, or exponent,** of a power is that number which indicates the degree or name of the power. It is placed at the right of the root near the top. Thus, 7^2 indicates the square of 7; $(\frac{3}{4})^3$ indicates the cube of $\frac{3}{4}$, &c. It is not necessary to put the index 1 to indicate the first power, and it is not done, except in developing principles.

Art. 425. *The product of two or more powers of a number is a power indicated by the sum of the indices of the factors.* Thus, $7^2 \times 7^3 = 7^{2+3}$, because $(7 \times 7) \times (7 \times 7 \times 7) = 7 \times 7 \times 7 \times 7 \times 7$, which is the fifth power of 7.

Art. 426. To find a required power of a number.

Rule.—*Use the number as a factor as many times as is indicated by the name of the power.*

NOTE.—Labor may sometimes be lessened by using the principle of Art. 425. Thus, the sixth power equals the product of the cube by itself, &c.

EXAMPLES FOR PRACTICE.

Find		Find	
1. 9^2	Ans. 81.	8. 99^3	
2. 10^2	Ans. 100.	9. 100^3	
3. 99^2	Ans. 9801.	10. 999^3	
4. 100^2	Ans. 10000.	11. 6^4	Ans. 1296.
5. 999^2		12. $(\frac{3}{4})^5$	Ans. $\frac{243}{512}$.
6. 9^3		13. $.05^6$	Ans. .000000015625.
7. 10^3		14. $(5^2)^3$	Ans. 5^6, or 15625.

Art. 427. *The square of a whole number has twice as many figures as the number, or one less than twice as many.*

NOTE.—For illustration, see Examples 1–5, above.

Art. 428. *The cube of a whole number cannot have more than three times as many figures as the number, nor fewer than three times as many less two.*

NOTE.—For illustration, see Examples 6–10, Art. 426.

Art. 429. *The square of any whole number is equal to the square of the local value of its first left-hand figure; plus twice the product of its value into that of the second; plus the square of that of the second; plus twice the product of that of the first and second into the local value of the third; plus the square of that of the third; plus twice the product of those of the first three into that of the fourth; plus the square of that of the fourth; and so on.*

ILLUSTRATION.

$$
\begin{array}{rlll}
325 = & & & 300 + 20 + 5 \\
\underline{325} = & & & \underline{300 + 20 + 5} \\
1625 = & & (300 \times 5) + & (20 \times 5) + 5^2 \\
6500 = & \quad (300 \times 20) + 20^2 & & + (20 \times 5) \\
97500 = 300^2 + & (300 \times 20) + & (300 \times 5) & \\
\hline
105625 = 300^2 + & 2(300 \times 20) + 20^2 + & 2(300 \times 5) + & 2(20 \times 5) + 5^2
\end{array}
$$

Art. 430. *The cube of any whole number is equal to the cube of the local value of its first left-hand figure; plus three times the square of that of the first into that of the second; plus three times that of the first into the square of that of the second; plus the cube of that of the second; plus three times the square of that of the sum of the first two into that of the third; plus three times that of the sum of the first two into the square of that of the third; plus the cube of that of the third; and so on.*

ILLUSTRATION.

$$
\begin{array}{lr}
256 = 16^2 = (100 + 120 + 36) = & 10^2 + 2(10 \times 6) + 6^2 \\
\underline{16} = & \underline{10 + 6} \\
1536 & (10^2 \times 6) + 2(10 \times 6^2) + 6^3 \\
2560 & 10^3 + 2(10^2 \times 6) + (10 \times 6^2) \\
\hline
4096 & 10^3 + 3(10^2 \times 6) + 3(10 \times 6^2) + 6^3
\end{array}
$$

EVOLUTION.

Art. 431. Evolution is the process of finding a root of a number. (See Art. 423.)

Roots are named from the number of times they are used as factors in producing the given power.

The **first root** of a number is the number itself.

The **second root,** or **square root,** of a number is one of its two equal component factors. Thus, 4 is the square root of 16.

The **third root,** or **cube root,** of a number is one of its three equal component factors. Thus, 4 is the cube root of 64.

To extract a certain root of a number is to resolve that number into as many equal factors as are indicated by the name of the root.

Rational roots are roots which can be exactly obtained.

Surd roots, or **surds,** are roots which cannot be exactly obtained.

Art. 432. The sign of evolution is either the **radical sign,** $\sqrt{}$, placed before the number whose root is indicated, or it is a fractional index, placed at the right of the number near its top. Thus, the square root of 5 is indicated either by $\sqrt{5}$, or by $5^{\frac{1}{2}}$, which signifies *the second root of the first power*, or *the first power of the second root;* the cube root of 8 is indicated either by $\sqrt[3]{8}$, or by $8^{\frac{1}{3}}$, which signifies *the third root of the first power*, or *the first power of the third root;* the fifth power of the fourth root of 16 is indicated either by $\sqrt[4]{16^5}$, or by $16^{\frac{5}{4}}$; the fourth root of the third power of $\frac{5}{8}$ is indicated either by $\sqrt[4]{(\frac{5}{8})^3}$, or by $(\frac{5}{8})^{\frac{3}{4}}$, &c. It is not necessary to indicate the first root of the first power, for example, $5^{\frac{1}{1}}$, and it is not done except in developing principles.

EXTRACTION OF THE SQUARE ROOT.

Art. 433. To extract the square root of a number is to resolve that number into two equal factors.

The multiplication table furnishes the following elements:—

ROOTS.—1,	2,	3,	4,	5,	6,	7,	8,	9,	10.
SQUARES.—1,	4,	9,	16,	25,	36,	49,	64,	81,	100.

CASE I.

Art. 434. To extract the square root of a whole number, or of a pure or mixed decimal.

Ex. 1. What is the square root of 576? Ans. 24.

WRITTEN PROCESS.

$$\begin{array}{r|l} & \overset{\frown}{5}\,\overset{\frown}{76}\,(24 \\ & 4 \\ \hline 44) & 176 \\ & \underline{176} \end{array}$$

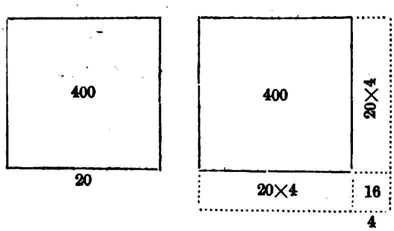
ANALYSIS OF WRITTEN PROCESS.

$$\begin{array}{lr|r|l} & & \overset{\frown}{5}\,\overset{\frown}{76} & 20 = 2 \text{ tens.} \\ \text{Square of 2 tens} & & = 400 & \ \ 4 = 4 \text{ units.} \\ \text{Twice } 20 = & 40 & & \\ 40 \times 4 = & 160 & 176 & 24 \text{ Answer.} \\ 4^2 = & 16 & \underline{176} & \end{array}$$

EXPLANATION.—Since 576 consists of three figures, its square root must consist of two figures. (See Art. 427.) Since the square of the tens of the root cannot be less than hundreds nor more than thousands, we must find the *tens* of the root from the 5 hundreds of the number. Now, the greatest number of tens whose square is contained in 576 is two tens, or 20, whose square is 400. This, taken from 576, leaves 176. This remainder must, by Art. 429, contain *twice the product of 20 by the units, plus the square of the units.* Therefore, to find the units, we must divide 176 by $2\times20=40$. The quotient is 4. Now, $4\times40+4^2=176$, which, subtracted from 176, leaves no remainder. In the written process we perform both multiplications at once, by uniting 4 with 40, making 44, before multiplying by 4.

ILLUSTRATION BY AREAS.

If 576 is a perfect square, it can be represented by the *area of a square* whose side is the square root of 576. Now, 20 units of length exhaust 400 units of area. This leaves 176 units of area to be added to two sides of the *approximate square*, 400, to make it the *required square*, 576. This implies *two side pieces*, each 20 units long, and *a square corner-piece*, with sides equal to their breadth. The whole



length of the two side pieces is 40. Dividing 176, *their approximate area*, by 40, *their length*, gives 4, *their breadth*. Now, the three added pieces are

$20 \times 4 = 80$, one side piece.
$20 \times 4 = 80$, the other side piece.
$4 \times 4 = \underline{16}$, the corner piece.

176, total; which makes the whole side of the required square $20 + 4 = 24$, and exhausts the area. If this effort had not exhausted the area, we should have had to apply the remainder to the sides of the area already found, and proceed as before.

Ex. 2. Find $\sqrt{9.3025}$. Ans. 3.05.

WRITTEN PROCESS.

		9.30 25(3.05
		9
$2 \times 3 = 6$:	60	30
$2 \times 30 = 60$:	605	3025
		3025

EXPLANATION.

Since this number has a decimal of four places, its square root has two decimal places, because a product has as many decimal places as are in both factors. Since the *tenths* of the root, squared, can produce no decimal beyond *hundredths*, that is, two decimal places, and since the *hundredths* of the root, squared, can produce no more than four decimal places, we must find the tenths of the root from the first two decimal places, and the hundredths of the root from the next two. Hence, we mark off the decimal *by twos* from the point toward the right, and extract the root as in Example 1. Now, we find that the divisor, 6, is contained in 3 tenths no tenths times. Write 0 in the tenths' place of the root, and bring down the next period, making 3025. Sixty tenths are contained in 302 thousandths 5 hundredths times. Write 5 in the hundredths' place of the root, also at the right of 60, making 605 the completed divisor. Finally, $5 \times 605 = 3025$, which exhausts the number.

Rule.—*From the place of the decimal point mark off the figures by twos: the last-formed period will have less than two figures when the number of figures is not divisible by 2.*

Find the greatest square in the left-hand period, write its root as the first figure of the answer, subtract the square from the period, and to the remainder annex the second period for a dividend.

At the left of this dividend write, as a trial divisor, double the root found; divide all but the right-hand figure of the dividend; annex the quotient both to the root found, as the next figure of the answer, and to the trial divisor.

Multiply this completed divisor by this last root figure; subtract the product, if possible, from the dividend, and to the remainder annex the next period for a new dividend. *Double the whole root found, for a* new trial divisor, *and proceed as before, till all the periods have been used.*

NOTE 1.—When the product is greater than the dividend, erase the root figure which produced it, and put a figure of less value in both the root and trial divisor, till the product is small enough for subtraction.

NOTE 2.—When the trial divisor is not contained in all of the dividend except its right-hand figure, annex a cipher both to the root and trial divisor, bring down the next period, and proceed as usual.

NOTE 3.—If, in marking off, the last decimal place has but one figure, annex a cipher to fill the period.

NOTE 4.—In *surds*, decimal periods of ciphers may be used to any sufficient degree of approximation.

NOTE 5.—Point off from the right of the root as many figures for decimals as there are decimal periods in the operation.

NOTE 6.—In large numbers, after finding more than the first half of the root, we can lessen labor by *constantly dividing the last remainder by the last divisor except its right-hand figure.* (See Example 3.)

NOTE 7.—*The square root of a perfect square may be found by resolving it into its prime factors, and finding the product of one of each two equal factors.* Thus, $1764 = (2 \times 2) \times (3 \times 3) \times (7 \times 7)$: hence $\sqrt{1764} = 2 \times 3 \times 7 = 42$.

EXAMPLES FOR PRACTICE.

3. Find the square root of 579. Ans. 24.062418+.

FULL PROCESS. (See Note 4.)

```
 579.0000000000(24.062418
 4
44)179
   176
4806)30000   (See Note 2.)
     28836
48122)116400
       96244
481244)2015600
        1924976
4812481)9062400
        4812481
48124828)424991900
         384998624
```

By Note 6.

```
579(24.062418
4
44)179
   176
4806)30000
     28836
48122)116400
       96244
48124)20156
      19248
        908
        481
        427
        384
```

NOTE.—In this contracted method we divide 20156 by 4812, then 908 by 481, then 427 by 48.

Find	Ans.	Find	Ans.
4. $\sqrt{1849.}$	43.	15. $1.5625^{\frac{1}{2}}$.	1.25
5. $\sqrt{3204100.}$	1790.	16. $\sqrt{.0000000576.}$	.00024.
6. $\sqrt{7569000000.}$	87000.	17. $.9^{\frac{1}{2}}$.	.9486+.
7. $\sqrt{93025.}$	305.	18. $\sqrt{.05.}$	.2236+.
8. $\sqrt{16564900.}$	4070.	19. $\sqrt{3.5.}$	1.8708+.
9. $\sqrt{4024036.}$	2006.	20. $\sqrt{.0007.}$	.02645+.
10. $\sqrt{5.4756.}$	2.34.	21. $\sqrt{189643.}$	
11. $2^{\frac{1}{2}}$.	1.414202+.	22. $64^{\frac{3}{2}}$.	512.
12. $8^{\frac{1}{2}}$.	2.828427+.	23. $\sqrt{9^5}$.	243.
13. $7^{\frac{1}{2}}$.	2.645751+.	24. $\sqrt{(\sqrt{923521})}$	31.
14. $10^{\frac{1}{2}}$.	3.162277+.	25. $\sqrt{10824.3216^{\frac{1}{2}}}$.	10.2.

CASE II.

Art. 435. To extract the square root of common fractions and of mixed numbers.

Ex. 1. Find the square root of $\frac{16}{25}$. Ans. $\frac{4}{5}$.

FIRST METHOD.

$$\sqrt{\tfrac{16}{25}} = \frac{\sqrt{16}}{\sqrt{25}} = \tfrac{4}{5}.$$

EXPLANATION.

The square root of $\frac{16}{25}$ is one of the two equal component factors of $\frac{16}{25}$. Therefore $\frac{16}{25}$ is the product of two equal fractions, and the numerator 16 is the square of the numerator of those fractions, and the denominator 25 is the square of the denominator of those fractions. Hence the numerator of those fractions is the square root of 16, and their denominator is the square root of 25.

SECOND METHOD.—$\frac{16}{25} = .64$: $\sqrt{.64} = .8 = \frac{8}{10} = \frac{4}{5}$.

Ex. 2. Find $\sqrt{5\frac{76}{100}}$. Ans. $2\frac{2}{5}$.

FIRST METHOD.—$5\frac{76}{100} = \frac{576}{100}$: $\sqrt{\frac{576}{100}} = \frac{24}{10} = 2\frac{4}{10} = 2\frac{2}{5}$.

SECOND METHOD.—$5\frac{76}{100} = 5.76$: $\sqrt{5.76} = 2.4 = 2\frac{2}{5}$.

Rules.—I. For a fraction. *Make the square root of the numerator a new numerator, and that of the denominator a new denominator;* or

Reduce it to a decimal, and find its square root.

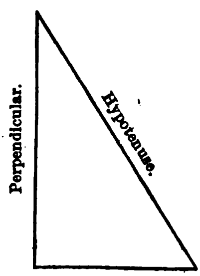
II. For a mixed number. *Reduce it to an improper fraction, or to a mixed decimal, and find its square root in this form.*

Examples for Practice.

3. Find $\sqrt{\frac{49}{81}}$. Ans. $\frac{7}{9}$.
4. Find $\sqrt{\frac{169}{225}}$. Ans. $\frac{13}{15}$.
5. Find $\sqrt{\frac{75}{192}}$. Ans. $\frac{5}{8}$.
6. Find $\sqrt{11\frac{1}{9}}$. Ans. $3\frac{1}{3}$.
7. Find $\sqrt{3\frac{1}{16}}$. Ans. $1\frac{3}{4}$.
8. Find $\sqrt{\frac{2}{3}}$. Ans. .816496+
9. Find $\sqrt{\frac{5}{8}}$.
10. Find $\sqrt{\frac{7}{10}}$.
11. Find $\sqrt{\frac{4}{5}}$.
12. Find $\sqrt{128\frac{3}{4}}$. Ans. 11.3468+

APPLICATIONS OF THE SQUARE ROOT.

Art. 436. **A plane triangle** is a plane figure bounded by three straight lines called its *sides*.

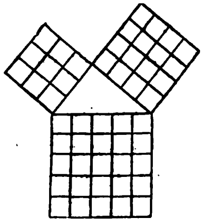


Base.

A **right-angled triangle** is a triangle, two of whose sides are perpendicular to each other. The **hypotenuse** is the side opposite to the right angle. It is the longest side. The **perpendicular sides** are the two sides which form the right angle. The **base** is that side on which the figure is supposed to rest. Any side of a triangle may be considered the base.

Art. 437. *The square of the hypotenuse is equal to the sum of the squares of the perpendicular sides.*

ILLUSTRATION.



On inspecting the figure in the margin, it is seen that the square

Of the side 3 units long	= 9	sq. units.
" " 4 " "	= 16	" "
And that the sum,	25	sq. units,

is equal to the square of the hypotenuse, which is 5 units long. This example is only one instance of a general truth, proved in Geometry.

Rules.—I. To find the hypotenuse. *Extract the square root of the sum of the squares of the other two sides.*

II. To find one perpendicular side. *Take the square of the given perpendicular from the square of the hypotenuse, and extract the square root of the remainder.*

III. To find the side of a square equal to a given area. *Find the square root of that area.*

Examples for Practice.

Find the hypotenuse of a triangle whose

1. Sides are 315 ft. and 420 ft. Ans. 525 ft.
2. Sides are 12 ft. 6 in. and 6 ft. 8 in. Ans. 14 ft. 2 in.

Find the other perpendicular of a triangle whose

3. Perp. is 24 ft. and hypotenuse 50 ft.
Ans. 43 ft. 10.35+ in.
4. Perp. is 15 ft. and hypotenuse 23 ft. 4 in.

Find the side of a square

5. Containing 1444804 sq. in. Ans. 100 ft. 2 in.
6. Containing 83 sq. ft.
7. Containing 40 A.
8. Two men start from the same point, and one travels due east 7.5 miles, the other due south 10 miles; how far are they apart?
9. The top of a pole standing 30 ft. from the shore of a river, is 80 ft. above the water, and 300 ft. in a straight line from the opposite shore; how wide is the river?
Ans. 259.13+ ft.
10. A ladder 45 ft. long reaches from a spot in the street 28 ft. up a house on one side, and 30 ft. up a house on the other side. What is the width of the street?
Ans. 68.768+ ft.
11. What is the distance from the lower corner to the opposite upper corner of a room 33 ft. long, 24 ft. wide, and 12 ft. high? Ans. 42.53+ ft.

Art. 438. *The areas of similar plane figures are to each other as the squares of the like dimensions of the figures.*

COROLLARY.—*The like dimensions of similar plane figures are to each other as the square roots of the areas of these figures.*

NOTE.—These propositions are proved in Geometry.

EXAMPLES FOR PRACTICE.

1. How much larger is a circle 12 inches in diameter than a circle 4 inches in diameter? Ans. 9 times.

2. How much larger is a lot 40 rods square than a lot 8 rods square? Ans. 25 times.

3. How much larger is a water-pipe 20 inches in diameter than one 6 inches in diameter? Ans. $11\frac{1}{9}$ times.

4. If a pipe $1\frac{1}{3}$ in. in diameter fill a cistern containing 48 gal. in a given time, what is the capacity of a cistern that a pipe $2\frac{1}{2}$ in. in diameter, will fill in the same time? Ans. $168\frac{3}{4}$ gal.

5. If a pipe $\frac{3}{4}$ in. in diameter fill a cistern in 8 hours, in what time will a pipe $2\frac{1}{4}$ in. in diameter fill the same cistern? Ans. $\frac{8}{9}$ of an hour.

6. A farmer has a field 60 rd. long and 48 rd. wide; what are the dimensions of a similar field containing 50 acres? Ans. 100 rd. by 80 rd.

7. How much larger is a hole bored by a 2-inch bit than one bored by a $\frac{1}{8}$-inch bit? Ans. 256 times.

8. If a pipe $1\frac{1}{2}$ inches in diameter will fill a cistern in 2 hr. 42 min., what must be the diameter of a pipe that will fill the same cistern in 2 hr. 8 min.?

EXTRACTION OF THE CUBE ROOT.

Art. 439. To extract the cube root of a number is to resolve that number into three equal factors.

CUBES WHOSE CUBE ROOTS ARE SINGLE INTEGRAL FIGURES.

CUBES.—	1,	8,	27,	64,	125,	216,	343,	512,	729.
ROOTS.—	1,	2,	3,	4,	5,	6,	7,	8,	9.

CASE I.

Art. 440. To extract the cube root of a whole number, or of a pure or mixed decimal.

Ex. 1. What is the cube root of 12167? Ans. 23.

WRITTEN PROCESS. EXPLANATION.

		12167(23
Cube of 2 tens =		8
$3 \times 20^2 =$	1200, trial d'r.	—
$3 \times 20 \times$ units $3 =$	180	4167
(Units $3)^2 =$	9	
	1389, true d'r.	4167

By Art. 428, this number has a cube root of two figures, namely, *units* and *tens*. The three right-hand figures 167, must contain the cube of the units of the root; the other two figures, 12, must contain the cube of the tens. Therefore, in general, *each group of three figures in order toward the left must furnish a figure of the cube root.* Hence, mark off the number from units, inclusive, into groups of three figures. By Art. 430, the two periods of this number, 12167, must contain

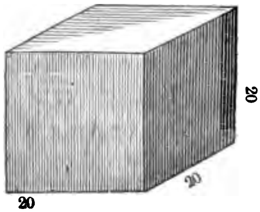
1. The cube of the tens of the root.
2. Three times the square of its tens × its units.
3. Three times its tens × its units × its units.
4. The cube of its units; = the sq. of its units × its units.

The greatest cube in the second period, 12, is 8, whose root is 2, which is the *tens* of the whole root. Taking 8 from 12, the remainder of the entire number is 4167, from which the *units* of the root must be obtained. Now, this 4167 must contain the last three of the above mentioned products, of each of which the units are one factor. Hence, dividing by one of the other factors, namely, *three times the square of the tens*, will *probably* give the other factor, namely, *the units*. On filling out the required products, they do not amount to more than the dividend: hence 3 is the units of the root. If the products filled out should in any case amount to more than the dividend, we must try a less figure for the root. In the foregoing written process we delayed multiplying by the units till we found the sum of the other factors, namely, 1389, thus multiplying them all at once; but each product could have been filled out by itself, thus:—

		12167	20
Cube of the tens $= 20^3 =$		8000	3
3×20^2	= 1200, trial divisor.		
$3 \times 20^2 \times 3$	= 3600	4167	23
$3 \times 20 \times 3^2$	= 540		
3^3	= 27	4167	

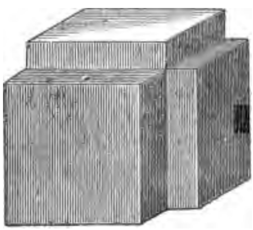
Illustration by Solids.

Fig. 1.



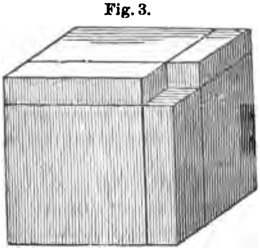
The number 2167 can be represented by a cubical block, whose volume is found by cubing the number representing the length of one edge, (see Art. 217,) and it is required to find that edge. The first part of the root—viz. 20—is an *approximation*, exhausting $20 \times 20 \times 20 = 8000$ units of volume. (See fig. 1.) This leaves 4167 units of volume to be added to the sides of the approximating cube, 8000, to make it equal to 2167, *still keeping it a cube.*

Fig. 2.



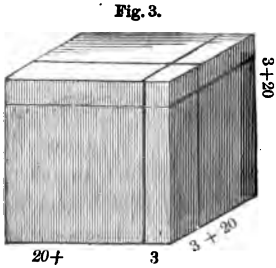
This implies three side slabs, each having one side $20 \times 20 = 400$. (See fig. 2.)

Fig. 3.



Also three corner-pieces, each 20 long, and having both breadth and thickness equal to the thickness of the slabs. (See fig. 3.)

Fig. 3.



Also one corner-cube, to fill the vacancy left by the three corner-pieces. (See fig. 4.)

Therefore, dividing the additional volume 4167 by the area of one side of all three slab-additions will give, either nearly or exactly, the thickness of these slabs. In the example it gives it exactly,—viz. 3. The added pieces, therefore, are in volume as follows:—

$20 \times 20 \times 3 =$ first side-slab,		$= 1200$
$20 \times 20 \times 3 =$ second "		$= 1200$
$20 \times 20 \times 3 =$ third "		$= 1200$
$20 \times 3 \times 3 =$ first corner-piece,		$= 180$
$20 \times 3 \times 3 =$ second "		$= 180$
$20 \times 3 \times 3 =$ third "		$= 180$
$3 \times 3 \times 3 =$ corner-cube.		$= 27$
	Total,	4167

Therefore, the approximate edge, 20, is, when completed, $20 + 3 = 23$. If 8000 and 4167 had not exhausted the volume, it is plain that the remaining volume should have been applied to the sides of the last formed approximate cube, in the same way as the first remainder was applied; and so with every successive remainder.

Again, if the number had been the pure decimal .012167, it is plain that its three equal component factors would be $.23 \times .23 \times .23$, and that the cube of .2 of the root would be in .012, and the cube of the .03 of the root would be in .000167. Hence the decimal should be marked off into groups of three figures from the decimal point toward the right. In a mixed decimal, the integral part must furnish the integral part of the root, and the decimal must furnish the decimal part of the root. Hence such a number should be marked off from the decimal point each way.

Rule.—*From the place of the decimal point mark off the figures by threes: the last-formed period will have less than three figures when the number of figures is not divisible by 3.*

Find the greatest cube in the left-hand period, write its cube root as the first figure of the answer, *subtract the cube from the period, and to the remainder annex the second period for a* dividend.

Square the root found, *annex two ciphers, and multiply the result by 3 for a* trial divisor.

Find how many times the trial divisor is contained in the dividend, and write the quotient as the next figure of the root.

To the trial divisor add three times the product of the former part of the root (with a cipher annexed) by the last figure of the root, and the square of the last figure.

Multiply this sum by the last figure of the root, subtract the product, if possible, from the dividend, and to the remainder annex the next period for a new dividend. *Proceed as before till all the periods have been used.*

NOTE 1.—If the product is greater than the dividend, erase the root figure that produced it, and with a figure of less value recalculate the additions to the trial divisor, till the product is small enough for subtraction.

NOTE 2.—If the trial divisor is not contained in the dividend, annex a cipher to the root, two ciphers to the trial divisor, bring down the next period to the right of the dividend, and proceed as usual.

NOTE 3.—Point off from the right of the root as many *figures* for decimals as there are *decimal periods* in the operation.

NOTE 4.—If, in marking off, the last decimal period has less than three figures, fill the period with ciphers.

NOTE 5.—In *surds*, decimal periods of ciphers may be used to any sufficient degree of approximation.

NOTE 6.—After getting any figure of the root by a trial divisor, we may, for a new dividend, subtract from all the periods that have been used the cube of all the root that has been found. This would be tedious in large operations.

NOTE 7.—The cube root of a *perfect cube* may be found by *resolving it into its prime factors, and finding the product of one of each three equal factors.* Thus, $2744 = (2 \times 2 \times 2) \times (7 \times 7 \times 7)$, and its cube root $= 2 \times 7 = 14$.

NOTE 8.—In finding cube root to a required number of decimal places, the following short methods may be used:—

After finding one more than half the required figures, annex one cipher to the last dividend, divide by the trial divisor, take the quotient as a new root figure, multiply the trial divisor by it, subtract the product, annex one cipher to the remainder, and so on. (See Ex. 28.) Or,

After finding one more than half the places, divide the last remainder continually by the last divisor, omitting each time two right-hand figures from the divisor, and one from the remainder. (See Ex. 29.)

EXAMPLES FOR PRACTICE.

Find	Ans.	Find	Ans.
2. $\sqrt[3]{42875.}$	35.	10. $\sqrt[3]{.091125.}$	.45.
3. $\sqrt[3]{1124864.}$	104.	11. $\sqrt[3]{.000000003375.}$	.0015.
4. $\sqrt[3]{2357947691.}$	1331.	12. $\sqrt[3]{2.}$	1.2599+.
5. $\sqrt[3]{389.017.}$	7.3.	13. $\sqrt[3]{3.}$	
6. $\sqrt[3]{1.860867.}$	1.23.	14. $\sqrt[3]{4.}$	1.5978+.
7. $\sqrt[3]{1012.048064.}$	10.04.	15. $\sqrt[3]{5.}$	
8. $\sqrt[3]{15.625.}$		16. $\sqrt[3]{.002.}$	
9. $\sqrt[3]{.015625.}$	.25.	17. $\sqrt[3]{.000006.}$	.01817+.

18. What is the cube root of 64000000? Ans. 400.
19. What is the cube root of 132651000000?
20. What is the cube root of 704969000?
21. What is the cube root of .300763? Ans. .67.
22. What is the cube root of .000001728? Ans. .012.
23. What is the cube root of 1860867?
24. What is the cube root of 389017000?
25. What is the cube root of .000000000000343?
26. Find the value of $(343)^{\frac{2}{3}}$. Ans. 49.
27. Find the value of $\sqrt[3]{9.1\dot{6}}$. Ans. 2.092+.
28. Find the cube root of .01525, correct to 7 decimal places. (Note 8.) Ans. .2479841±.
29. Find the cube root of $.8\dot{3}$, correct to 5 decimal places. (Note 8.) Ans. .94103±.
30. Find the cube root of 12, correct to 7 decimal places.
31. Find the value of $\sqrt[3]{144^2}$ to 4 places. Ans. 27.47+.
32. Find the value of $5^{\frac{4}{3}}$ to 5 places. Ans. 8.5499.

CASE II.

Art. 441. To extract the cube root of a common fraction, or of a mixed number.

PROPOSITION.—*The cube root of a fraction is the cube root of its numerator divided by the cube root of its denominator.*

DEMONSTRATION.—A fraction is the product of three equal factors, each of which is its cube root. The numerator of the fraction is the product of three equal numerators, and its denominator is the product of three equal denominators. Therefore the cube root of the fraction is a fraction whose numerator is the cube root of its numerator, and whose denominator is the cube root of its denominator.

ILLUSTRATION.—The fraction $\frac{27}{64}$ is the product of three equal fractions, each of which is $\frac{3}{4}$. Hence its numerator 27 is the product of the three numerators, and its denominator 64 is the product of the three denominators. Hence $\frac{3}{4}$ is the cube root of $\frac{27}{64}$.

Rules.—I. For a fraction. *Make the cube root of the numerator a new numerator, and the cube root of the denominator a new denominator.* Or,

Reduce it to a decimal, and find its cube root.

II. For a mixed number. *Reduce it to an improper fraction, or to a mixed decimal, and find its cube root in this form.*

Examples for Practice.

1. $\sqrt[3]{\frac{64}{125}} = \frac{4}{5}$ or 8.	7. $\sqrt[3]{166\frac{3}{5}} =$	13. $\sqrt[3]{\frac{2197}{39304}} = \frac{13}{34}$.
2. $\sqrt[3]{3\frac{3}{8}} = 1\frac{1}{2}$ or 1.5.	8. $\sqrt[3]{\frac{3}{4}} = .908+$.	14. $\sqrt[3]{6\frac{1}{4}} = 1.84+$.
3. $\sqrt[3]{4\frac{17}{27}} = 1\frac{2}{3}$.	9. $\sqrt[3]{\frac{2}{3}} = .86+$.	15. $\sqrt[3]{5\frac{1}{2}} =$
4. $\sqrt[3]{15\frac{5}{8}} =$	10. $\sqrt[3]{\frac{5}{6}} = .94+$.	16. $\sqrt[3]{132\frac{3}{4}} =$
5. $\sqrt[3]{13\frac{103}{125}} = 2\frac{2}{5}$.	11. $\sqrt[3]{\frac{4}{5}} =$	17. $\sqrt[3]{296\frac{8}{27}} = 6\frac{2}{3}$.
6. $\sqrt[3]{5\frac{23}{64}} =$	12. $\sqrt[3]{\frac{7}{8}} =$	18. $\sqrt[3]{8\frac{1}{4}} =$

APPLICATIONS OF THE CUBE ROOT.

Art. 442. *The volumes of similar solids are to each other as the cubes of their corresponding linear dimensions.*

COROLLARY. *The corresponding linear dimensions of similar solids are to each other as the cube roots of the volumes of those solids.*

NOTE.—These propositions are proved in geometry.

Examples for Practice.

1. If a cannon ball 3 inches in diameter, weighs 10 pounds, what is the weight of a ball 9 inches in diameter?
Ans. 270 lb.

2. If a cannon ball 6 inches in diameter, weighs 81 pounds, what must be the diameter of a similar ball to weigh $\frac{8}{9}$ of a pound?
Ans. $1\frac{1}{3}$ inches.

3. What must be the edge of a cubical bin that shall contain as many bushels, as a bin 10 ft. 5 in. long, 5 ft. 4 in. wide, and 2 ft. 3 in. deep?
Ans. 5 ft.

4. If a stack of hay 12 feet in diameter contain 5 tons, will be the diameter of a similar stack to contain 10
Ans. 15.1+ft.

5. If a man 5 ft. 4 in. in height, weigh 125 pounds, how tall is that man whose weight is 216 pounds?

Ans. 6 ft. $4\frac{4}{5}$ in.

6. How many square feet in the surface of a cubical block of marble containing 125 cu. ft.? Ans. 150.

7. How many times can a keg 12 in. in diameter at the bung, be filled from a similarly shaped barrel whose bung-diameter is 2 ft. 6 in.? Ans. $15\frac{5}{8}$.

8. How many $\frac{1}{2}$-inch balls will weigh as much as a 4-inch ball of the same material? Ans. 512.

9. What must be the edge of a cubical bin that will contain 100 bushels?

10. What is the difference between $\frac{1}{2}$ of a cu. ft. and $\frac{1}{2}$ of a foot cubed?

11. What are the linear dimensions of a rectangular bin, whose capacity is 61440 cu. in.; the length, breadth, and depth being to each other as 10, 4, and 3?

Ans. 3 ft. 4 in., 2 ft. 8 in., and 2 ft.

12. A and B bought a conical stack of hay 15 feet high, containing 6 tons. How much must A take from the top of the stack that he may have one-half of it? Ans. 11.9+ft.

13. Separate 10290 into three factors that shall be to each other as 2, 3, and 5. Ans. 14, 21, and 35.

EXTRACTION OF ANY ROOT.

Art. 443. To extract any root of a number.

Rule I. *Begin as many columns of numbers as there are units in the index of the root, by writing the given number as the head of the right-hand column, and ciphers as the head of the others.*

Beginning at the place of the decimal point, mark off the number into as many periods as possible of as many figures each as there are units in the index of the root.

Find the required root of the greatest perfect power, of the

given degree, in the left-hand period, and write it as the first figure of the answer.

Add this figure to the term of the left-hand column, for its next term. Multiply this sum by the root-figure; add the product to the term in the second column; multiply this sum by the same root-figure, and add the product to the term in the third column; and so proceed till you place the last product under that period in the right-hand column from which the root-figure was found. Subtract the last product from the period above it.

Add the same root-figure to the last term of the first column, for its next term. Multiply the sum by the root-figure; add the product to the last term of the second column; and so proceed, ending in the last column but one. Repeat the process, ending each time one column farther to the left, till you at last merely begin it in the left-hand column.

Annex one cipher to the last term in the first column, two ciphers to that in the second, and so on; and to that in the last column annex the next period for a dividend. Divide this by the last term of the previous column, as a trial divisor, (making allowance for completing it,) for the next figure of the root, with which do as with the first figure, and so on till all the periods have been used.

NOTE 1.—If the exponent of the required root is a composite number, that root may be found by extracting successively the roots expressed by the factors of the exponent. Thus, the fourth root may be found by extracting *the square root of the square root*; the sixth root may be found by extracting *the square root of the cube root*, or *the cube root of the square root;* the eighth root, by extracting *the square root of the square root of the square root*, or *the square root of the fourth root*, or *the fourth root of the square root*, &c. Thus, if the sixth power of 5 is the product of $5 \times 5 \times 5 \times 5 \times 5 \times 5$, its sixth root, 5, may be found by either finding the cube root, which is one of three equal factors, 5×5, and then the square root of that, or by first finding the square root, which is one of two equal factors, $5 \times 5 \times 5$, and then the cube root of that. To illustrate this by indices, $5^{\frac{6}{6}} = 5^{\frac{6}{3} \times \frac{1}{2}}$, or $5^{\frac{6}{2} \times \frac{1}{3}}$.

NOTE 2.—This method of extracting any root of a number is called "Horner's Method," because it was invented by Mr. Horner, of Bath, England. It is only an application of his general method of resolving *algebraic* equations. (See Higher Algebra.)

Examples for Practice.

Ex. 1. Find the fifth root of 45435424. Ans. 34.

Written Process.

o	o	o	o	45435424(34
3	9	27	81	243
3	9	27	81	21135424
3	18	81	324	21135424
6	27	108	4050000	
3	27	162	1233856	
9	54	270000	5283856	
3	36	38464		
12	9000	308464		
3	616			
150	9616			
4				
154				

By this rule find

2. The square root of 1849. Ans. 43.
3. The cube root of 140608.
4. The fourth root of 16777216. Ans. 64.
5. The fifth root of $.41\dot{6}$. Ans. .83+.
6. The sixth root of 191102976.

CHAPTER XIX.

SERIES.

Art. 444. **A series,** or **progression,** is a succession of numbers, each of which is formed from the preceding, according to a definite law.

The **terms** of a series are the numbers which compose it.

The **extremes** of a series are its first and last terms.

The **means** are all the terms except the first and last.

An **ascending,** or **increasing, series** is a series composed of terms, each of which is larger than the preceding term.

A **descending,** or **decreasing, series** is a series composed of terms, each of which is smaller than the preceding term.

Art. 445. In reference to the constancy of the rates between any two consecutive terms, series are *variable* or *constant.*

A **variable series** is a series whose ratio between different consecutive terms is not the same.

A **constant series** is a series whose ratio between different consecutive terms is the same.

In reference to the kind of their ratios, series are *arithmetical,* or *geometrical.*

An **arithmetical series** is a series whose terms increase or decrease by differences either constant or variable.

A **geometrical series** is a series whose terms increase or decrease by multipliers either constant or variable.

NOTE.—The discussion of variable series belongs to Algebra, and its applications to the Differential and Integral Calculus. The only kind usually treated of in Arithmetic is the constant.

ARITMETICAL OR EQUIDIFFERENT SERIES.

Art. 446. An **arithmetical,** or **equidifferent, series** is a series in which the difference of the consecutive terms is constant. Thus, 1, 5, 9, 13, &c., is an increasiug arithmetical series, whose constant difference is 4: and 17, 12, 7, 2 is a decreasing arithmetical series, whose constant difference is 5.

CALCULATIONS IN EQUIDIFFERENT SERIES.

Art. 447. The quantities considered in calculating equidifferent series are the *first term, difference, number of terms, last term,* and *sum of the terms.*

In the notations of equidifferent series, a signifies the first term, d the difference, n the number of terms, l the last term, and s the sum of the terms. The sign $+$ before d signifies that the difference is to be added, and the series is increasing; the sign $-$ before d signifies that the difference is to be subtracted, and the series is decreasing; and the double sign $\pm$ before d signifies that the difference is either additive or subtractive.

CASE I.

Art. 448. To find one extreme, when the other extreme, the difference, and the number of terms are known.

Of the series a, $a \pm d$, $a \pm 2d$, $a \pm 3d$, &c., the n^{th} term is, manifestly, $a \pm (n - 1)\,d$. Hence $l = a \pm (n - 1)\,d$.

Rule.—*Multiply the difference by the number of times less one, and add the product to the given extreme, if it is the less, but subtract the product from the given extreme, if it is the greater.*

NOTE.—If in the equation $l = a \pm (n - 1)\,d$ we suppose $d = 0$, then l will equal a; that is, the series will merely be a series of equal numbers.

Examples for Practice.

1. Find the 16th term of 2, 8, 14, etc. Ans. 92.
2. Find the 40th term of 3, 10, 17, etc.
3. Find the 50th term of 250, 247, 244, etc. Ans. 103.
4. Find the 81st term of $12\frac{1}{2}$, $18\frac{5}{8}$, $25\frac{1}{8}$, etc.
5. Find the 24th term of 88.5, 83.9, 81.6, etc.
6. A man bought 100 head of sheep, at 43.5 cts. for the first sheep, 47.2 cts. for the second, 50.9 cts. for the third, and so on; what did the last sheep cost?

Case II.

Art. 449. To find the sum of an equidifferent series.

If to the equation $s = a + (a \pm d) + (a \pm 2d) + \ldots + l$
we add $s = l + (l \mp d) + (l \mp 2d) + \ldots + a$
we have $2s = (a+l) + (a+l) + (a+l) + \ldots + (a+l)$
or $2s = n\,(a+l)$: and $s = \frac{1}{2}\,(n\,(a+l))$ Hence, half the sum of the extremes is the average of the terms.

Rule.—*Multiply half the sum of the extremes by the number of terms, or the sum of the extremes by half the number of terms, or multiply the sum of the extremes by the number of terms and divide the product by 2.*

Examples for Practice.

What is the sum

1. Of 30 terms of 2, 8, 14, etc.? Ans. 2650.
2. Of 100 terms of $2\frac{1}{2}$, $3\frac{3}{4}$, 5, etc.?
3. Of 121 terms of 100, $99\frac{1}{5}$, $98\frac{2}{5}$, etc.? Ans. 11858.
4. Of 52 terms of $\frac{3}{5}$, $1\frac{4}{15}$, $1\frac{14}{15}$, etc.? Ans. $915\frac{1}{5}$.
5. Of 11 terms of .6, 1.3, 2, etc.? Ans. 45.1.
6. Of 41 terms of 75.2, 73.5, 71.8, etc.?
7. How far would you travel, starting from a basket 2 yd. from the first apple, and carrying singly to the basket 100 apples 2 yd. apart in a straight line?

Ans. 11 mi. 3 fur. 32 rd. 4 yd.

8. The first term of an arithmetical progression is 250, the last term 103, and the number of terms 50; what is the sum of all the terms?

CASE III.

Art. 450. To find the difference, when the extremes and number of terms are known.

Since $l = a \pm (n - 1)\, d$, it is plain that $l - a = \pm\, (n - 1)\, d$, and $d = \pm\, ((l - a) \div (n - 1))$. Hence the

Rule.—*Divide the difference of the extremes by the number of terms less one.*

EXAMPLES FOR PRACTICE.

1. The first term is 2, the last term is 74, and the number of terms 25; what is the common difference?

2. The first term is 730, the last term 2, and the number of terms 365; what is the common difference? Ans. 2.

3. $750 in 50 years, at simple interest, amounts to $3000; what is the annual interest? Ans. $45.

4. If the extremes are $2\frac{3}{5}$ and $26\frac{29}{40}$, and the number of terms 40, what is the common difference? Ans. $\frac{5}{8}$.

5. The extremes are 1.5 and 255.5, and the number of terms 101; what is the common difference?

6. The extremes are 0 and $7\frac{1}{2}$, and the number of terms 16; what is the common difference?

PROBLEM.

Art. 451. To insert a given number of means between two given numbers.

Since $n =$ two more than the number of means, $n - 1 =$ one more than the number of means. Therefore $d = (l - a)$ divided by one more than the number of means. Hence the

Rule.—*Divide the difference of the extremes by the number of means plus one, and use the quotient as a difference.*

1. Insert 4 arithmetical means between 5 and 25.

2. Insert 3 arithmetical means between $\frac{1}{3}$ and $\frac{1}{2}$.
   Ans. $\frac{3}{8}$, $\frac{5}{12}$, $\frac{11}{24}$.

3. Insert 5 arithmetical means between 5 and 15.
   Ans. $6\frac{2}{3}$, $8\frac{1}{3}$, 10, $11\frac{2}{3}$, $13\frac{1}{3}$.

CASE IV.

Art. 452. To find the number of terms, when the extremes and difference are known.

Since $l - a = (n - 1)\ d$, it is plain that $(l - a) \div \quad = n - 1$, and that $((l - a) \div d) + 1 = n$. Hence the

Rule.—*Divide the difference of the extremes by the common difference, and add 1 to the quotient.*

EXAMPLES FOR PRACTICE.

1. The extremes are 2 and 30, and the common difference $\frac{4}{5}$; what is the number of terms? Ans. 36.
2. The annual interest of $450 is $27, in how many years, at simple interest, will it amount to $990? Ans. 20.
3. If the extremes are 203 and 8, and the common difference 5, what is the number of terms?
4. The extremes are 6 and 54.75, and the common difference 3.25; what is the number of terms?

GEOMETRICAL OR EQUIRATIONAL SERIES.

Art. 453. A **geometrical,** or **equirational, series** is a series in which the ratio of the consecutive terms is constant. Thus, 1, 2, 4, 8, 16, &c., is a geometrical series increasing by a constant multiplier, 2, called the *ratio:* and 16, 8, 4, 2, 1 is a geometrical series decreasing by the constant multiplier, $\frac{1}{2}$, called the *ratio.*

NOTE.—If we represent a series as a series of ratios in usual form, thus:—1 : 2 : : 2 : 4 : : 4 : 8 : : 8 : 16, &c., and represent the ratio in the usual way, namely, the division of the antecedent by the consequent, the ratio would be less than 1 in an increasing series and greater than 1 in a decreasing series. It has been customary, however, to consider the ratio of a series as a *multiplier.*

The quantities considered in calculating equirational series are the *first term, ratio, number of terms, last term,* and *sum of terms.* In formulas r signifies ratio, and a, l, n, s signify the same as in formulas of equidifferent series. (Art. *448.)*

CALCULATIONS IN EQUIRATIONAL SERIES.

CASE I.

Art. 454. To find one extreme, when the other extreme, the ratio, and the number of terms are known.

Of the series $a\,r$, $a\,r^2$, $a\,r^3$, &c., the n^{th} term is, manifestly, $a\,r^{n-1}$. Hence, $l = a\,r^{n-1}$.

Rule.—*Multiply the given extreme by that power of the ratio whose exponent is one less than the number of terms.*

NOTE 1.—If the given extreme is the smaller, reckon the ratio as in an ascending series; if the given extreme is the larger, reckon the ratio as in a descending series.

NOTE 2.—If $r = 1$, all the terms will be equal.

EXAMPLES FOR PRACTICE.

1. The first term of a geometrical series is 5, the ratio 6, and the number of terms 7; what is the last term?
Ans. 233280.

2. If the first term is 9, the ratio $\frac{2}{3}$, and the number of terms 6, what is the last term? Ans. $1\frac{5}{27}$.

3. A drover bought 11 horses, paying \$3 for the first, \$6 for the second, \$12 for the third, and so on; what did he pay for the last one? Ans. \$3072.

4. The last term of an equirational series is 312500, the ratio 5, and the number of terms 8; what is the first term?
Ans. 4.

CASE II.

Art. 455. To find the sum of an equirational series.

If $s = a + a\,r + a\,r^2 + \ldots a\,r^{n-1}$,
then $r\,s = \quad a\,r + a\,r^2 + \ldots a\,r^{n-1} + a\,r^n$. Subtracting, we have $(r-1)\,s = a\,(r^n - 1)$. Hence $s = a\,(r^n - 1) \div (r - 1)$. Hence the

Rule.—*Multiply the last term by the ratio, find the difference between this product and the first term, and divide the result by the difference between the ratio and 1.*

NOTE.—If a series is infinite, the less extreme is 0.

Reduce it to a decimal, and find its cube root.

II. For a mixed number. *Reduce it to an improper fraction, or to a mixed decimal, and find its cube root in this form.*

EXAMPLES FOR PRACTICE.

1. $\sqrt[3]{\frac{64}{125}} = \frac{4}{5}$ or 8.	7. $\sqrt[3]{166\frac{3}{8}} =$	13. $\sqrt[3]{\frac{2197}{39304}} = \frac{13}{34}$.
2. $\sqrt[3]{3\frac{3}{8}} = 1\frac{1}{2}$ or 1.5.	8. $\sqrt[3]{\frac{3}{4}} = .908+$.	14. $\sqrt[3]{6\frac{1}{4}} = 1.84+$.
3. $\sqrt[3]{4\frac{17}{27}} = 1\frac{2}{3}$.	9. $\sqrt[3]{\frac{2}{3}} = .86+$.	15. $\sqrt[3]{5\frac{1}{2}} =$
4. $\sqrt[3]{15\frac{5}{8}} =$	10. $\sqrt[3]{\frac{5}{6}} = .94+$.	16. $\sqrt[3]{132\frac{3}{4}} =$
5. $\sqrt[3]{13\frac{103}{125}} = 2\frac{2}{5}$.	11. $\sqrt[3]{\frac{4}{5}} =$	17. $\sqrt[3]{296\frac{8}{27}} = 6\frac{2}{3}$.
6. $\sqrt[3]{5\frac{23}{64}} =$	12. $\sqrt[3]{\frac{7}{8}} =$	18. $\sqrt[3]{8\frac{1}{4}} =$

APPLICATIONS OF THE CUBE ROOT.

Art. 442. *The volumes of similar solids are to each other as the cubes of their corresponding linear dimensions.*

COROLLARY. *The corresponding linear dimensions of similar solids are to each other as the cube roots of the volumes of those solids.*

NOTE.—These propositions are proved in geometry.

EXAMPLES FOR PRACTICE.

1. If a cannon ball 3 inches in diameter, weighs 10 pounds, what is the weight of a ball 9 inches in diameter?

   Ans. 270 lb.

2. If a cannon ball 6 inches in diameter, weighs 81 pounds, what must be the diameter of a similar ball to weigh $\frac{8}{9}$ of a pound?

   Ans. $1\frac{1}{3}$ inches.

3. What must be the edge of a cubical bin that shall contain as many bushels, as a bin 10 ft. 5 in. long, 5 ft. 4 in. wide, and 2 ft. 3 in. deep?

   Ans. 5 ft.

4. If a stack of hay 12 feet in diameter contain 5 tons, what will be the diameter of a similar stack to contain 10 tons?

   Ans. 15.1+ft.

5. If a man 5 ft. 4 in. in height, weigh 125 pounds, how tall is that man whose weight is 216 pounds?

Ans. 6 ft. $4\frac{4}{5}$ in.

6. How many square feet in the surface of a cubical block of marble containing 125 cu. ft.? Ans. 150.

7. How many times can a keg 12 in. in diameter at the bung, be filled from a similarly shaped barrel whose bung-diameter is 2 ft. 6 in.? Ans. $15\frac{5}{8}$.

8. How many $\frac{1}{2}$-inch balls will weigh as much as a 4-inch ball of the same material? Ans. 512.

9. What must be the edge of a cubical bin that will contain 100 bushels?

10. What is the difference between $\frac{1}{2}$ of a cu. ft. and $\frac{1}{2}$ of a foot cubed?

11. What are the linear dimensions of a rectangular bin, whose capacity is 61440 cu. in.; the length, breadth, and depth being to each other as 10, 4, and 3?

Ans. 3 ft. 4 in., 2 ft. 8 in., and 2 ft.

12. A and B bought a conical stack of hay 15 feet high, containing 6 tons. How much must A take from the top of the stack that he may have one-half of it? Ans. 11.9+ft.

13. Separate 10290 into three factors that shall be to each other as 2, 3, and 5. Ans. 14, 21, and 35.

EXTRACTION OF ANY ROOT.

Art. 443. To extract any root of a number.

Rule I. *Begin as many columns of numbers as there are units in the index of the root, by writing the given number as the head of the right-hand column, and ciphers as the head of the others.*

Beginning at the place of the decimal point, mark off the number into as many periods as possible of as many figures each as there are units in the index of the root.

Find the required root of the greatest perfect power, of the

EXPLANATION.—The first \$300 amounts in 4 yr. to \$372; the second \$300 in 3 yr. to \$354; the third \$300 in 2 yr. to \$336; the fourth \$300 in 1 yr. to \$318; and the fifth \$300 is due without interest. These numbers form an arithmetical series, of which the annuity, \$300, is the first term, the common difference is the interest of the annuity for 1 yr., \$18, the number of terms is the number of years, 5, and the sum of the terms is the amount.

Rule.—*Make the interest of one period the common difference, the annuity the first term, the number of periods the number of terms; then find its last term by* **Art. 448,** *and sum by* **Art. 449.**

2. What is the amount of an annuity of \$500 for 8 years, at 6% simple interest? Ans. \$4840.

3. What is the amount of an annuity of \$200 for 10 years, at 6% simple interest?

4. A worked for B one year, at \$60 per month, payable monthly; provided nothing was paid until the expiration of the year, what was then due him with simple interest, at 8%? Ans. \$746.40.

5. What is the amount of an annuity of \$160 for $2\frac{1}{2}$ years, payable quarterly, at $1\frac{1}{4}$% per quarter? Ans. \$1690.

CASE II.

Art. 459. To find the present worth of an annuity, at simple interest.

Rule.—*Divide the amount of the annuity by the amount of \$1, for the given time and rate.*

1. What is the present worth of an annuity of \$400 for 6 years, at 6% simple interest? Ans. \$2029.41+.

2. B leased a lot for 5 years for \$1000 a year; what sum, at 8% simple interest, will discharge the obligation? Ans. \$4142.85+.

3. What is the present worth of an annuity of \$300 for 21 years, at 6% simple interest?

ANNUITIES AT COMPOUND INTEREST.

CASE I.

Art. 459. To find the amount of an annuity at compound interest.

Ex. 1. What is the amount of an annuity of $300 for 5 years, at 6% compound interest? Ans. $1691.13.

EXPLANATION.—The fifth $300 is due without interest; the fourth amounts in 1 yr. to $\$300 \times 1.06$; the third in 2 yr. to $\$300 \times 1.06 \times 1.06 = \$300 \times 1.06^2$; the second in 3 yr. to $\$300 \times 1.06 \times 1.06 \times 1.06 = \$300 \times 1.06^3$, &c. These numbers form a geometrical series, of which the annuity, $300, is the first term, the ratio is the amount of $1 for 1 yr., $1.06, the number of years is the number of terms, and the sum of the series is the amount of the annuity.

Rule.—*Make the amount of one unit of money for one period of time the ratio, the annuity the first term, and the number of periods the number of terms; then find its last term by* **Art. 454,** *and its sum by* **Art. 455.**

NOTE.—The *last term* equals the annuity multiplied by the amount of $1 for the given time and rate per cent., taken from the table, Art. 385.

2. What is the amount of an annuity of $200 for 4 years, at 6% compound interest? Ans. $874.92⅓.

3. What is the amount of an annuity of $300 for 25 years, at 6% compound interest? Ans. $16459.35.

4. A father deposited in a Savings Bank for his son, on his twelfth birthday $100, and the same amount on each subsequent birthday; how much did this amount to when the son was 21 years old, at 5% compound interest? Ans. $1257.77.

5. A man expends $200 annually for tobacco and drinks. If he should dispense with these and lend the $200, at 6% compound interest, how much would it amount to in 40 years? Ans. $30952.39.

6. Find the amount of an annuity of $500 for 5 years at 6% compound interest.

Case II.

Art. 460. To find the present worth of an annuity, at compound interest.

Rule.—*Divide the amount of the annuity by the compound amount of $1 for the given time and rate per cent. taken from the table,* **Art. 385.**

1. What is the present worth of an annuity of $400 for 6 years, at 6% compound interest? Ans. $1966.92+.

2. What is the present worth of an annuity of $500 for 10 years, at 5% compound interest?

3. An annuity of $300 for 20 years is in reversion 8 years. What is its present value, compound interest at 5%?
Ans. $2530.47+.

4. An annuity of $200 for 15 years is in reversion 10 years. What is its present worth, at 6% compound interest?

PERMUTATIONS.

Art. 461. **Permutations** are changes in the arrangement of a given number of things.

Case I.

Art. 462. To find the possible number of arrangements of a given number of things, taken all at a time.

The two letters b and c can be arranged, as usually written, in $1 \times 2 = 2$ ways; thus, $b\,c$ and $c\,b$. The three letters b, c, and d can be arranged in $1 \times 2 \times 3 = 6$ ways; thus, $b\,c\,d$, $b\,d\,c$, $c\,b\,d$; $c\,d\,b$, $d\,b\,c$, $d\,c\,b$; that is, the new letter, d, takes three positions in each of the two arrangements $b\,c$ and $c\,b$. A fourth letter, f, can take four positions in each of the six arrangements of $b\,c\,d$; hence four things can make $1 \times 2 \times 3 \times 4 = 24$ arrangements.

Rule.—*Find the product of as many numbers of the natural series 1, 2, 3, &c., as there are objects to be arranged.*

EXAMPLES FOR PRACTICE.

In how many ways can the following be arranged?

1. Five recitations by a class? Ans. 120.
2. Six teachers in the same school? Ans. 720.
3. Seven musical notes? Ans. 5040.
4. Eight pupils in a class?
5. The nine digits?
6. In how many different ways can 10 cars be arranged in making up a train?

CASE II.

Art. 463. To find the possible number of arrangements of a given number of things, taken a given less number at a time.

Each of the four letters *b, c, d, f*, prefixed to *each* of the other three, would make three arrangements *by twos;* in all, $4 \times 3 = 12$ arrangements *by twos.* Again, if to each of the 12 arrangements *by twos*, each of the remaining two letters be prefixed, $4 \times 3 \times 2 = 24$ arrangements *by threes* will result.

Rule.—*Multiply together the numbers of the decreasing natural series, whose greatest term is the number of objects, and whose number of terms is the number of objects taken at a time.*

EXAMPLES FOR PRACTICE.

1. How many whole numbers can be expressed by the nine digits, taken three at a time? Ans. 504.
2. How many, taken six at a time? Ans. 90720.
3. How many, taken five at a time?
4. In how many ways can the letters of the alphabet be written, taken two at a time? Three at a time?

COMBINATIONS.

Art. 464. Combinations are groups so formed from the members of a given number of things that all the members of one group are not the same as those of another.

Art. 465. To find the number of possible combinations of a given number of things, taken a given number at a time.

Rule.—*Divide the number of possible permutations of the whole number, taken the given number at a time, by the number of possible permutations of as many things as are taken at a time.*

EXAMPLES FOR PRACTICE.

1. How many combinations can be made up of the nine digits, three in a set? Ans. 84.
2. How many, six in a set? Ans. 126.
3. Of ten letters, five in a set? Ans. 252.
4. Of seven musical notes, three in a set? Ans. 35.
5. Of eight persons, four in a set?
6. Of the letters in *Pittsburgh*, six in a set?
7. Of twelve letters, seven in a set?

CHAPTER XX.

MENSURATION.

Art. 466. Mensuration is the art of measuring magnitudes.

In reference to the kind of magnitude, mensuration is of *lines*, *surfaces*, and *solids*.

The measurement of lines is linear measure. (See Arts. 205–211.)

MENSURATION OF SURFACES.

Art. 467. In mensuration of surface, (see Arts. 212–216,) the definitions are given, and the processes are demonstrated by geometry.

A **figure** is a surface limited by a line or lines.

In reference to their limiting lines, figures are *rectilinear*, or *curvilinear*.

A **rectilinear figure** is a figure limited by straight lines.

A **curvilinear figure** is a figure limited by a curved line, or curved lines.

Regular surfaces are either *plane* or *curved*.

A **plane surface** is a surface such that a straight line between any two of its points lies wholly in the surface.

A **curved surface** is a surface which is neither plane, nor composed of plane surfaces.

The usual unit of measure for both plane and curved surfaces is the area of a plane surface, called a *square*.

The **base** of a figure is the side on which it is supposed to rest.

Art. 468. In rectilinear figures

A **polygon** is any rectilinear figure.

A *regular polygon* is a polygon of equal sides and equal angles.

An *irregular polygon* is a polygon of unequal sides and unequal angles.

A **triangle** is a plane figure of three sides.

An *equilateral triangle* is a triangle having three equal sides.

An *isosceles triangle* is a triangle having two equal sides.

A *scalene triangle* is a triangle having no equal sides.

The **altitude** of a triangle is the shortest distance from the vertex of one of its angles to the opposite side taken as a base.

A **quadrilateral** is a plane figure of four sides.

A **parallelogram** is a quadrilateral having its opposite sides parallel and equal.

A **rectangle** is a quadrilateral whose angles are right angles.

A **square** is a rectangle of equal sides.

A **rhombus,** or **rhomb,** is a parallelogram of equal sides and oblique angles.

A **rhomboid** is a parallelogram whose opposite sides only are equal, and whose angles are oblique.

A **trapezium** is a quadrilateral having no parallel sides.

A **trapezoid** is a quadrilateral having only two parallel sides.

A **pentagon** is a polygon of five sides; a **hexagon,** of six; a **heptagon,** of seven; an **octagon,** of eight; a **nonagon,** of nine; a **decagon,** of ten; an **undecagon,** of eleven; a **duodecagon,** of twelve.

A **diagonal** is a line which joins two angles of a figure which are not adjacent.

The **perimeter** of a figure is the sum of its sides.

A line is *inscribed in a circle* when its ends are in the circumference. A polygon is inscribed in a circle when its sides are inscribed. A circle is inscribed in a polygon when the

circumference touches each side. In the last two cases the outer figure *circumscribes* the inner.

The **radius** of a circle is a straight line whose ends are at the centre and circumference.

The **diameter** of a circle is a straight line passing through the centre and having its ends in the circumference.

The **centre of a regular polygon** is the centre of either the inscribed or circumscribed circles.

The **radius of a regular polygon** is the radius of the circumscribed circle.

The **apothem** of a regular polygon is the radius of the inscribed circle.

The **altitude** of a parallelogram or trapezoid is the perpendicular distance between two parallel sides when one of them is taken as a base.

Art. 469. In mensuration of surfaces and solids the factors must express linear units of the same kind. The product expresses, in surface measure, square units of the same name as the linear, and, in solid measure, cubic units of the same name as the linear.

Art. 470. To find the area of a parallelogram.

Rule.—*Multiply the base by the altitude.*

1. How many sq. yd. in a sidewalk 18 yd. long and 9 ft. wide? Ans. 54.

2. What is the area of a parallelogram whose base is 21 ft. and altitude 15 ft.?

3. How many more acres in a piece of land 120 rods square, than in a rectangular piece 32 rods long and 30 rods wide? Ans. 84 A.

Art. 471. To find the area of a triangle.

NOTE.—A triangle is half of a parallelogram of equal base and altitude.

Rules.—I. When base and altitude are known. *Multiply the base by the altitude and take half the product.*

II. When three sides are known. *From half the sum of the sides take each side separately; multiply together the half sum and three remainders, and find the square root of the product.*

1. The base of a triangle is 24 ft., and its altitude 16 ft. what is its area? Ans. 192 sq. ft.

2. What is the area of a triangle whose base is 74 rd. and altitude 40 rd.?

3. What is the area of a triangle whose sides are, respectively, 10, 16, and 20 rods? Ans. 79.24+ sq. rd.

4. What is the area of a triangle whose sides are, respectively, 18, 20, and 24 inches?

Art. 473. To find the area of a trapezoid.

Rule.—*Multiply half the sum of the parallel sides by the altitude.*

1. How many square feet in a board 18 ft. long, 18 in. wide at one end, and 14 in. at the other? Ans. 48.

2. What is the area of a piece of sheet-tin 32 in. long, 18 in. wide at one end, and 12 in. at the other?

3. How many acres in a field 54 rd. long, 32 rd. wide at one end, and 40 rd. at the other?

Art. 474. To find the area of an irregular polygon.

Rule.—*Separate the polygon into triangles by diagonals, and find the sum of their areas.*

Art. 475. To find the area of a regular polygon of more than four sides.

Rule.—*Find half the product of apothem and perimeter.*

1. What is the area of a trapezoid whose diagonal is 40 ft., and whose sides are, on one side of it 20 ft. and 30 ft., and on the other 25 ft. each? Ans. 59.04+ sq. ft.

2. What is the area of a regular pentagon, each of whose sides is 15 ft., and apothem 10.32 ft.? Ans. 387 sq. in.

3. What is the area of a regular hexagon, each of whose sides is 10 in., and apothem 8.66 in.?

Art. 476. To find the circumference of a circle.

Rule.—*Multiply the diameter by 3.1416.*

1. What is the circumference of a circle whose diameter is 5 feet?

2. What is the circumference of a circle whose diameter is 22.5 inches?

3. What must be the length of a fence that will enclose a circular lot 40 rd. in diameter?

4. If the radius of a circle is 12.5 ft., what is its circumference? Ans. 78.54 ft.

Art. 477. To find the diameter of a circle.

Rule.—I. *Divide the circumference by 3.1416.*

1. What is the diameter of a circle whose circumference is 157.08 feet?

2. What is the diameter of a log whose circumference is 8.6394 feet? Ans. 2 ft. 9 in.

Art. 478. To find the area of a circle.

Rules.—I. *Multiply the square of the diameter by .7854.*

II. *Multiply the square of the radius by 3.1416.*

III. *Multiply the diameter by the circumference, and find one-fourth of the product.*

1. What is the area of a circle whose diameter is 15 inches? Ans. 176.715 sq. in.

2. What is the area of a circle whose radius is 20 feet?

3. What is the area of a circle whose circumference is 31.416 rods?

MENSURATION OF SOLIDS.

Art. 479. A **solid** is a magnitude which has length, breadth, and thickness.

A **prism** is a solid whose ends are equal polygons, and whose sides are parallelograms.

A **cylinder** is a regular solid whose ends are equal and parallel circles.

The **altitude** of a prism is the perpendicular distance between the planes of its ends or bases.

A **pyramid** is a solid having a regular polygon for its base, and for its sides triangles which meet in a common point, called a *vertex.*

A **cone** is a solid having a circular base, and whose convex surface tapers uniformly to a point, called a vertex.

The **altitude** of a pyramid or cone is the perpendicular distance from its vertex to its base.

The **slant height** of a pyramid is the distance from the vertex to the middle of one side of the base.

The **slant height** of a cone is the distance from the vertex to the circumference of the base.

The **frustrum** of a pyramid or cone is the part that remains after the top is cut off by a plane parallel with the base.

A **sphere** is a solid, every part of whose surface is equally distant from a point within, called the centre.

The **diameter** of a sphere is a straight line passing through the centre, and terminating in the surface.

The **radius** is half the diameter.

Art. 480. To find the surface of a prism or cylinder.

Rule.—*Multiply the perimeter of its end by its altitude, and to the product add the area of both ends.*

1. How many square feet in the surface of a marble slab 6 ft. long, 2 ft. wide, and 5 in. thick?

2. What is the convex surface of a cylinder 18 ft. long, and 5 ft. in circumference? Ans. 90 sq. ft.

3. What is the surface of a cylindrical log whose diameter is 20 in. and length 10 ft.? Ans. 56.723+ sq. ft.

Art. 481. To find the volume of a prism or cylinder.

Rule.—*Multiply the area of its base by its altitude.*

1. What is the capacity of a cylindrical vessel 20 in. in diameter and 30 in. deep? Ans. 5.454+ cu. ft.

2. What is the volume of a cylinder 5 in. in diameter and 15 in. long?

3. What is the volume of a triangular prism 20 in. long, and whose sides are 4 in., 6 in., and 8 in.?
Ans. 232.37+ cu. in.

Art. 482. To find the convex surface of a pyramid or cone.

Rule.—*Multiply the perimeter of the base by half the slant height.*

NOTE.—To find the whole surface, add the area of the base.

1. What is the convex surface and whole surface of right pyramid whose slant height is 115 feet, and whose base is 320 ft. square? Ans. Conv. surf. 73600 sq. ft.; whole surf. 176000 sq. ft.

2. Of a right pyramid whose slant height is 32 ft., and whose base is 60 ft. square?

3. Of a cone whose slant height is 100 ft., and the diameter of whose base is 10 ft.? Ans. 1570.8 sq. ft.; 1649.34 sq. ft.

Art. 483. To find the volume of a pyramid or cone.

Rule.—*Multiply the area of the base by one-third of the altitude.*

1. What is the volume of a pyramid 180 ft. high, and 200 ft. square at the base? Ans. 2400000 cu. ft.

2. What is the volume of a triangular pyramid 300 ft. high, and the sides of whose base are each 60 ft.?

3. How many cubic inches in a conical glass 4 in. in diameter, and 6 in. deep? Ans. 25.1328 cu. in.

Art. 484. To find the surface of a frustrum.

Rule.—*Multiply the sum of the perimeters of the two ends by half the slant height; to the product add the areas of the ends.*

1. How many square feet in the whole surface of the frustrum of a cone whose top diameter is 6 ft., bottom diameter 10 ft., and whose slant height is 20 ft.? Ans. 609.4704.

2. What is the surface of a frustrum of a pyramid whose slant height is 8 in.; upper base 4 in. square; lower base 6 in. square? Ans. 200 sq. in.

Art. 485. To find the volume of a frustrum.

Rule.—*Find the square root of the product of the areas of the two ends; to this root add the two areas, and multiply the sum by one-third of the altitude of the frustrum.*

NOTE.—The square root of the product of the areas of the two ends is a mean proportion between them.

1. What is the volume of a log 30 ft. long, the diameter of one end being 3 feet, and of the other 2 feet?

Ans. 149.226 cu. ft.

2. What is the volume of the frustrum of a square pyramid the sides of whose bases are 3 and 4 feet, respectively, and whose altitude is 15 feet? Ans. 185 cu. ft.

Art. 486. To find the surface of a sphere.

Rule.—*Multiply the square of the diameter by 3.1416.*

1. What is the surface of a sphere 30 in. in diameter?

2. How many square miles in the surface of the earth, supposing its diameter to be 8000 miles?

Art. 486. To find the volume of a sphere.

Rule.—*Multiply the cube of the diameter by .5236.*

1. What is the volume of a spherical shot 10 inches in diameter?

2. What is the solidity of a sphere 12 inches in diameter?

MISCELLANEOUS EXAMPLES.

1. From 294 English trillions subtract 435 French trillions, and write the result in words by both methods

2. Express in Roman characters 97, 243, 755, 105, 2979, and 1874.

3. What number increased by $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{8}$ of itself equals 315?

4. What number diminished by $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{6}$ of itself equals 315?

5. Find the greatest common divisor of $\frac{18}{35}$, $\frac{45}{56}$. and $\frac{63}{80}$.

6. What number is that, from which, if $10\frac{3}{4}$ be subtracted, $\frac{3}{5}$ of the remainder is $142\frac{1}{2}$?

7. How many days from Jan. 10, 1874, to July 4, 1876? Ans. 906.

8. 159 is $37\frac{1}{2}\%$ of what number?

9. Reduce .619047 to a common fraction. Ans. $\frac{13}{21}$.

10. How many pounds Avoirdupois are there in 1225 pounds Troy?

11. What is the interest of $1360 for 5 yr. 10 mo. 12 da., at 7% per annum?

12. In what time will $1296, at 6%, gain $668.10?

13. If $\frac{3}{5}$ of the cost of an article equals $\frac{4}{7}$ of the selling price, what is the gain per cent.? Ans. 5%.

14. A, B, and C entered into partnership and gained $2175; A had $1500 invested 8 mo., B $1800 invested 6 mo., and C $2300 invested 9 mo.; what was the gain of each? Ans. A's, $600; B's, $540; C's, $1035.

15. From $\frac{3}{5}$ of $\frac{2}{7}$ of $\frac{1}{8}$ of 112 miles subtract .32 of 1 mi. 90 rd. Ans. 1 mi. 316 rd. 13 ft. $2\frac{2}{5}$ in.

16. If $6\frac{3}{5}$ yd. of carpet, $1\frac{3}{4}$ yd. wide, cost $17.50, how much should $8\frac{3}{4}$ yd., $2\frac{1}{4}$ yd. wide, cost?

17. A is worth $6435, which is $\frac{5}{7}$ of what B is worth; how much are they both worth?

18. What is the difference between the interest and the discount of $720 for 5 yr. 7 mo. 20 da., at 6%?

19. What must I ask for cloth that cost \$4.50 per yard, that I may abate 10% from the asking price, and still make 18%? Ans. \$5.90.

20. How many cubic yards in a room 21 ft. long, 15 ft. wide, and 12 ft. high?

21. In digging a cellar 24 ft. long, and 5 ft. deep, 80 cu. yd. of earth were removed; how wide is it?

22. A man spent 80% of 90% of 26% of his money, and had \$568.96 remaining; how much had he at first?

Ans. \$700.

23. Find the least common multiple of $\frac{15}{22}$, $\frac{27}{44}$, and $\frac{45}{77}$.

24. If \$380 in 3 yr. 4 mo. 12 da. gain \$76.76, what is the rate per cent.?

25. What is the hour, when the time past noon is equal to $\frac{1}{5}$ of the time to midnight? Ans. 2 o'clock.

26. A slab of marble 6.7 ft. long, and 3.4 ft. wide, contains 8.5425 cu. ft., how thick is it? Ans. $4\frac{1}{2}$ in.

27. What is the area of a triangle whose sides are 18, 22, and 40 ft. respectively?

28. Reduce $.1\dot{7}1428\dot{5}$ to a common fraction. Ans. $\frac{6}{35}$.

29. If I buy stocks at 10% discount, and sell at 10% premium, what per cent. do I gain on my money?

Ans. $22\frac{2}{9}$%.

30. What is the length of a cubical bin that will contain 1728 bushels of wheat? Ans. 12 ft. 1.08+ in.

31. If 54 men in 63 days of 10 hours each, dig a trench $67\frac{1}{2}$ yd. long, $5\frac{1}{4}$ yd. wide, and $2\frac{3}{4}$ yd. deep, how many hours a day must 62 men work, to dig a trench $46\frac{1}{2}$ yd. long, $3\frac{2}{3}$ yd. wide, and $2\frac{1}{3}$ yd. deep, in $19\frac{1}{4}$ days? Ans. 12.

32. I invest and sell at a loss of 25%; I invest the proceeds again, and sell at a gain of 25%; do I gain or lose on the two speculations, and how many per cent.?

Ans. Lose $6\frac{1}{4}$%.

33. What is the least sum of money for which I could purchase a number of hogs at $\$20\frac{4}{7}$, a number of cows at $\$42\frac{2}{3}$, or a number of horses at \$72, and what number of *each could I* purchase for that sum?

34. A pole whose length was 164 feet, was in the air and water; $\frac{3}{5}$ of the length in the air equaled $\frac{4}{7}$ of the length in the water. What length was in the air and water respectively? Ans. In air, 80 ft.; in water, 84 ft.

35. If the bank discount on a 90-day note at 9% is \$11.16, what is the face?

36. A, B, and C do a piece of work for \$860; A sends 8 men 5 days, B sends 7 men 6 days, and C sends 9 men 10 days; what should each receive?

37. A stock-dealer bought a number of sheep for \$5640, 45 of them having died, he sold $\frac{5}{8}$ of the remainder for \$4700, which was \$225 more than cost; how many sheep did he buy? Ans. 940.

38. I invest and sell at 12% gain; invest the proceeds and sell at an advance of 15%; invest the proceeds again, and sell at a loss of 25%, and quit with \$1254.80; what was my capital at first? Ans. \$1300.

39. What part of 20 is $\frac{5}{6}$ of 10?

40. What is the difference between the true and bank discount of \$1440 for 90 days, at 6%? Ans. \$1.04.

41. If $\frac{5}{8}$ of the selling price is 10% less than the cost, what is the gain per cent.?

42. A man owes \$5400, one-third of which was due in 1 yr., one-third in 2 yr., and the remainder in 3 yr.; what sum will discharge the debt now, money worth 6%?

Ans. \$4830.68.

43. If I pay 108% for U. S. bonds bearing $7\frac{3}{10}$%, what rate of interest do I receive? Ans. $6\frac{41}{54}$%.

44. What must I pay for stocks bearing 6% that I may invest my money at 7%? Ans. $85\frac{5}{7}$%.

45. How many cast-iron balls 4, 6, or 8 inches in diameter, can be placed in a cubical vessel whose edge is 2 feet; and how many gallons of water will it contain after it is filled with balls? Ans. 216 4-inch balls; 64 6-inch balls; 27 8-inch balls; and 28.509 gallons of water.

46. By the rule of the U. S. Courts, what was due on the following note, Dec. 20, 1873?

4000\frac{00}{100}$. PITTSBURGH, *Jan. 10, 1872.*

For value received, six months after date, I promise to pay Geo Aiken, or order, four thousand dollars, with interest from date, at 6%. WALTER S. DOBSON.

INDORSEMENTS.—July 2, 1872, $1000; Dec. 10, 1872, $100; May 10, 1873, $1500; Sept. 25, 1873, $600.

47. If a man begins at 25 years of age, and deposits annually $200 in a Savings Bank which pays 6% compound interest, how much will the bank owe him at 50 years of age? Ans. $10972.90+.

48. If a pipe $\frac{3}{4}$ of an inch in diameter fill a cask in 5 minutes, how long will it take a pipe $\frac{1}{2}$ inch in diameter to fill it? Ans. $11\frac{1}{4}$ min.

49. What is the length of a rafter, the roof being 8 ft. high and the building 30 ft. wide?

50. When gold is at $111\frac{1}{2}$%, what is the value, in currency, of $1250 in gold?

51. When gold is at $112\frac{1}{2}$%, what is the value of a dollar in currency? Ans. $.88$\frac{8}{9}$.

52. An agent received $25.20 for selling a house at a commission of 3%; what did he get for the house?

53. A commission merchant sold a consignment of 1400 barrels of pork, at $12.50 per barrel. After deducting $73.24 for transportation, $19.50 for storage, and his commission, he remits to his employer $16777.26 as the net proceeds of the sale. What was his rate of commission? Ans. $3\frac{3}{8}$%.

54. A broker bought School Bonds at 95%, and sold at $97\frac{1}{2}$%. His profits were $200; what was the amount of his purchase? Ans. $8000.

55. What cost 28 shares of R. R. stock, par value $100 per share, at $6\frac{1}{4}$% premium, brokerage $\frac{3}{8}$%?

56. Find the diagonal of a square field containing 9 A. Ans. 53.66+ rd.

57. How many acres in a square field whose diagonal is 40 rd.? Ans. 5 A.

58. I wish to obtain from a bank $7874 for 60 days, at 9%; for what sum must I give my note?

59. Says A to B, $\frac{3}{4}$ of my age equals $\frac{4}{5}$ of yours, and the sum of our ages is 124 years. Required the age of each.

Ans. A's age 64 years, B's age 60 years.

60. A person being asked the time of day, said "$\frac{3}{4}$ of the time past noon equals $\frac{4}{5}$ of the time from now to midnight;" what was the time? Ans. 20 minutes past 5 o'clock.

61. What per cent. advance on stocks paying 9% must I pay to invest at 7%? Ans. $28\frac{4}{7}$%.

62. What is the discount on $1480.39, for 2 yr. 9 mo. 15 da., at 6%?

63. By buying stocks at $2\frac{3}{4}$% discount, and selling at $3\frac{1}{2}$% premium, I gained $150. What sum did I invest?

Ans. $2334.

64. What will be the cost of paving a court 120 ft. square, a walk 8 ft. wide around the whole being paved with flag-stones, at $.63 a square yard, and the remainder with bricks at $.36 a square yard? Ans. $683.52.

65. If $\frac{4}{5}$ of the selling price is 20% more than cost, what is the gain per cent.? Ans. 50%.

66. What principal will, in 3 yr. 10 mo. 9 da., at 7%, amount to $4572.30?

67. Bought stocks at 15% discount; sold out at a gain of 40%, realizing $3213; what sum did I invest?

Ans. $2295.

68. Bought 16 shares of bank stock, par value $100 per share, at a premium of $2\frac{1}{2}$%, and sold the same at a discount of $3\frac{2}{5}$%; how much did I lose?

69. The sum of two numbers is 230, and their difference is .217; what are the numbers?

70. If $360 gain $54.72 in 2 yr. 6 mo. 12 da., in what time will $600 gain $122?

71. What must I pay for a \$1000 mortgage bearing 6%, that I may receive 8% for my money?

72. Bought a house for \$2400; how much must I ask for it that I may fall 20% from my asking price, and still make 25% on my investment? Ans. \$3750.

73. Which is the better investment—School bonds at 102% bearing 7% interest in currency, or U. S. bonds at 115% bearing 6% interest in gold, gold being $112\frac{1}{2}$%?

74. Having sold a house and lot on 3% commission, I invested the proceeds in a farm, after deducting my purchase commission of 2%. My whole commission was \$530; what was the cost of the farm? Ans. \$10282.

75. I have a circular piece of ground whose diameter is 100 ft. If I pay \$.18 per square yard for constructing a gravel walk 10 ft. wide around the whole, and \$.45 per square yard for paving the remainder, what will be the entire cost? Ans. \$347.93+.

76. If A can do a piece of work in 5 days, B in 8 days, and C in 10 days, how long will it take them all working together to do it? Ans. $2\frac{6}{17}$ days.

77. Divide \$900 between A and B, so that $\frac{2}{3}$ of A's share + \$80, shall equal $\frac{3}{4}$ of B's share.

Ans. A's, \$420; B's, \$480.

78. A boy after spending $\frac{2}{3}$ of all his money, and $\frac{4}{5}$ of the remainder, had only \$.56 remaining; how much had he at first? Ans. \$3.60.

79. A man had 10% of his sheep stolen, 25% of the remainder killed by dogs, and 10% of what then remained died, when he had 792 head left; how many sheep had he at first?

80. What is the compound interest of \$324 for 5 years, at 6%?

81. What will be the cost of 9 boards, each 22 feet long, 16 inches wide at one end, and 20 inches at the other, at *\$34.50 per M.?* Ans. \$10.2465.

82. A tract of land 95 rd. long contains 38 acres; how wide is it?

83. What is the radius of a circular tract of land containing 2 acres? Ans. 10 rd. 1 ft. 6.3+ in

84. If New York is 74° 3″ west, and Chicago is 87° 35″ west, what is the time at Chicago, when it is 12 o'clock 10 min. $6\frac{2}{5}$ sec. at New York?

85. A broker received $7227 to invest in P. R. R. stock; after reserving his brokerage, $\frac{3}{8}$%, required the amount invested and the brokerage.

86. How much larger is a lot 30 rods square than a circular lot whose diameter is 30 rods?

87. How many solid feet in a pile of wood 16 ft. 6′ long, 3 ft. 10′ wide, and 4 ft. 4′ high?

88. I sold a lot of grain through a broker who charged me $2\frac{3}{4}$%. My commission was $4\frac{1}{8}$%, and after paying my broker I had $165 remaining; what was the value of the grain sold? Ans. $12000.

89. An old lady bought eggs at the rate of 3 for 2 cents, and sold them at the rate of 2 for 3 cents; what per cent. did she gain? Ans. 125%.

90. Received an invoice of Port wine, 20% of which was lost by breakage; at what per cent. above cost must he sell the remainder, to clear 25% on the invoice? Ans. $56\frac{1}{4}$%.

91. I invested $13128 in U. S. bonds at 108%, paying $1\frac{2}{5}$% brokerage, and afterward sold at 115%, paying $1\frac{3}{4}$% brokerage; what did I gain? Ans. $462.

92. What length of board 10 in. wide will make a square foot?

93. How many bushels will a box contain whose inside dimensions are 6 ft. 4 in. long, 4 ft. 3 in. wide, and 3 ft. 6 in. deep?

94. How many feet of inch lumber in the box?

95. The interest of A's, B's, and C's fortunes, for 3 yr. 10 mo. 10 da., at 6%, is 10378.66\frac{2}{3}$. What is the fortune of each, if $\frac{2}{3}$ of A's fortune equals $\frac{3}{4}$ of B's, and $\frac{3}{5}$ of B's equals $\frac{8}{9}$ of C's? Ans. A's $18000, B's $16000, C's $10800.

96. What is the true balance of the following account at settlement, July 1, 1871? Ans. $1410.71+.

Dr. S. C. WEBER IN ACCT. CUR. WITH HARRIS CRAWFORD. Cr.

1871.		$	ct.	1871.		$	ct.
Jan. 14.	To Mdse. (at 2 mo.)	600		May 6.	By Cash.	800	
Feb. 5.	" " (at 4 mo.)	800		May 8.	" Mdse. (at 2 mo.)	600	
Mar. 2.	" " (at 3 mo.)	900		June 4.	" " (at 2 mo.)	300	
Apr. 12.	" " (at 3 mo.)	1200		June 20.	" Cash.	400	

97. A and B can do a piece of work in 10 days, A and C in 12 days, and B and C in 15 days; in how many days can they all do it? Ans. 8 days.

98. In what time can each do it?

99. A carpenter engaged to work for 40 days, on condition that he should receive $2.85 for each day he worked, and forfeit $.85 for each day he was idle. At the expiration of the time he received $69.60; how many days was he idle?

Ans. 12.

100. In a battle 6% of the army were killed, and 5% of the remainder were captured. The difference between the killed and captured was 169; how many men were in the army, how many were captured, and how many were killed?

101. Sold 14000 pounds of wool, at $.62 a pound; invested the proceeds in sheep, reserving my commissions, 4% for selling, and $1\frac{1}{2}$% for buying; what was my entire commission? Ans. $470.34+.

102. A man being asked the time of day, replied that $\frac{2}{3}$ of the time past 8 o'clock A. M., equaled $\frac{2}{5}$ of the time to midnight; what was the time? Ans. 2 o'clock P. M.

103. What length of twine will wind spirally around a cylinder that is 68 ft. long and 4 ft. 3 in. in circumference, provided it passes around once every 5 ft. 8 in. Ans. 85 ft.

104. A man purchased 500 acres of land at $60 per acre, and agreed to pay for it in 6 annual instalments. If he withhold the instalments till 8 years from the time the first *was due, what* sum will then discharge the debt, interest 6%?